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# ON THE COMPUTATIONAL POWER OF AUTOMATA WITH TIME OR SPACE BOUNDED BY ACKERMANN'S OR SUPEREXPONENTIAL FUNCTIONS

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**Abstract.** In this paper, we investigate the classes of the functions computed by automata introduced in [11] with time or space bounded by Ackermann's or superexponential functions.

## 1. Introduction

The computational power of the ideal computers without limit in running time and storage space is very strong. It is well known that the functions computed by such machines are just the partial recursive functions whatever mathematical models we may select [2, 10, 11]. In contrast with the ideal situation, the storage capacity of a practical computer is bounded by a constant, so that the computational power drops sharply and the set of all the functions computed by such machines is a proper sublcass of the elementary functions [11]. Therefore, one of the important problems in automata theory and complexity theory is the computational power of time- (or space-) bounded automata. Ritchie [6] investigated the computational power of tape-bounded Turing machines, and showed that $F_0$, $F_1$, $F_2$, ... constitute a hierarchy of the elementary functions, where $F_0$ is the set of all functions computed by finite automata and $F_i$ the set of all total functions computed by Turing machines with tape bounded by functions in $F_{i-1}$, $i \geqslant 1$. Cobham [1] characterized Grzegorczyk's function classes from the viewpoint of computational complexity and observed that for any $n \geqslant 3$, $\mathscr{E}^n$ is the set of all total functions computed by Turing machines with time or space bounded by functions in $\mathscr{E}^n$ (or with step- or tape-length-counting functions belonging to $\mathscr{E}^n$). For the primitive recursive word functions similar results were given in [4]. Meyer and Ritchie [5] generalized Grzegorczyk's hierarchy of the primitive recursive functions to the general recursive functions from the viewpoint of computational complexity; in addition, they pointed out that for any $n \geqslant 3$, $\mathscr{E}^n$ is the set of all total functions computed by Turing machines with time bounded by $f_n^{(c)}(\max(x_1, \ldots, x_n))$, $c = 0, 1, \ldots$, where

$$f^{(0)}(x) = x, \qquad f^{(y+1)}(x) = f(f^{(y)}(x)), \qquad f_3(x) = 2^x, \qquad f_{n+1}(x) = f_n^{(x)}(1), \quad n > 3.$$

For unlimited register machines (URM) [10] it is easy to derive that for any $n \geqslant 3$, $\mathscr{E}^n$ is the set of all total functions computed by URM (singleton alphabet) with time bounded by functions in $\mathscr{E}^n$ [9]. All the works enumerated above regard the computational complexity class as a total function class, therefore partial functions are excluded. This is not natural! Given a Turing machine (or another automaton) $M$ and a time- (or space-) limit function $g$, for any initial input of $M$, as the running time (or storage capacity) exceeds the limit given by both $g$ and the initial input, we regard naturally the corresponding value of the function computed by $M$ as to be undefined. Therefore, given $g$, for any $M$ there is a function which is computed by $M$ with time (or space) bounded by $g$. In this paper, computational complexity classes contain total and partial functions as well.

Recall some definitions and notations in [11]. An *automaton* $M$ is specified by the following features:

(1) The *internal alphabet* $\mathscr{A}_0$ and the $i$th *external alphabet* $\mathscr{A}_i$, $i = 1, \ldots, m$, $m > 0$ being the number of tapes;

(2) The *capacity* $q > 0$, the $i$th *input-length* $l_i \geqslant 0$ and the $i$th *ouptut-length* $r_i \geqslant 0$, $i = 1, \ldots, m$;

(3) The *internal state transformation* $\rho$ (a single-valued mapping from $\mathscr{S}_0$ to $\mathscr{S}_0$), the $i$th *input characteristic function* $\eta_i$ (a single-valued mapping from $\mathscr{S}_0$ to $\{0, l_i\}$), and the $i$th *output function* $\lambda_i$ (a single-valued mapping from $\mathscr{S}_0$ to $\{\Lambda\} \cup W_{r_i}(\mathscr{A}_i)$), $i = 1, \ldots, m$, where $\mathscr{S}_0 = W_q(\mathscr{A}_0) \times \bigcup_{i=0}^{l_1} W_i(\mathscr{A}_1) \times \cdots \times \bigcup_{i=0}^{l_m} W_i(\mathscr{A}_m)$, $W_i(\mathscr{A})$ denotes the set of all words of length $i$ on alphabet $\mathscr{A}$, and $\Lambda$ the empty word.

Denote the set of all words on $\mathscr{A}$ by $W(\mathscr{A})$. Let $\mathscr{S}_M = W_q(\mathscr{A}_0) \times W(\mathscr{A}_1) \times \cdots \times W(\mathscr{A}_m)$, and refer to any entry $\gamma = (x_0, \ldots, x_m)$ in $\mathscr{S}_M$ as a (*total*) *state of* $M$, to $x_0$ as the *internal state*, to $x_i$ as the $i$th *external state*, $i = 1, \ldots, m$, to $\gamma_0 = (x_0, L_1 x_1, \ldots, L_m x_m)$ as the *scan state of* $\gamma$, and to $L_{l_i} x_i$ as the $i$th *external scan state*, $i = 1, \ldots, m$, where $L_l x$, i.e. $L(l, x)$, is the longest left segment of $x$ of length $\leqslant l$. For any $\gamma$ in $\mathscr{S}_M$, let

$$\xi_M(\gamma) = (\rho(\gamma_0), R(-\eta_1(\gamma_0), x_1)\lambda_1(\gamma_0), \ldots, R(-\eta_m(\gamma_0), x_m)\lambda_m(\gamma_0)),$$

where $\gamma_0$ is the scan state of $\gamma$, and $R(-l, x)$, i.e. $R_{-l}x$, the right segment of $x$ with $L_l x R_l x = x$. Denote $\xi_M^0(\gamma) = \gamma$, $\xi_M^{n+1}(\gamma) = \xi_M(\xi_M^n(\gamma))$. A state $\gamma$ of $M$ is said *to be a halting state* if $\gamma = \xi_M(\gamma)$ and $\lambda_i(\gamma_0) = \Lambda$, $i = 1, \ldots, m$, $\gamma_0$ being the scan state of $\gamma$. For any $\gamma$ in $\mathscr{S}_M$, if $\min\{k \mid k \geqslant 0$ and $\xi_M^k(\gamma)$ is a halting state$\}$ exists and has the value $n$, then we define $\mathrm{Im}_M \gamma$ as $\xi_M^n(\gamma)$, otherwise $\mathrm{Im}_M \gamma$ is undefined. In this paper, we call $M$ *to be unary* if $|\mathscr{A}_1| = \cdots = |\mathscr{A}_m| = 1$, $|\mathscr{A}|$ being the number of symbols in $\mathscr{A}$. We use $M$, with or without superscript and/or subscript, to denote an automaton, and $m$, $\mathscr{A}_j$, etc., with or without superscript and/or subscript, to denote the corresponding number of tapes, $j$th external alphabet, etc.

These automata as defined above include Shepherdson and Sturgis's URM($\mathscr{A}$), without macroinstructions $O(n)$ and $C(n, k)$, as a special case in which all the external alphabets are $\mathscr{A}$ and all the input- and output-length are 1. These automata are more like modern computers than Turing machines. Therefore, we use them as a mathematical model of computers.

In this paper, we first investigate the classes of functions computed by automata with time or space bounded by Ackermann's functions. We then discuss the inclusion relations between the time complexity classes and the space complexity classes and between the complexity classes and the unary complexity classes, and show that the set of all functions computed by automata bounded by Ackermann's functions is PR($\theta$) (the set of all functions which are primitive recursive in $\theta$) and these complexity classes constitute a hierarchy of PR($\theta$). Finally, complexity classes with superexponential time and space bounds are also discussed.

## 2. Computation with time bounded by Ackermann's functions

In this paper, we fix an infinite alphabet $a_1, a_2, \ldots, a_n, \ldots$. For any automaton, without loss of generality, we assume that each external alphabet of the automaton consists of some initial segment of the above infinite alphabet.

Denote $N = \{0, 1, 2, \ldots\}$. For any positive integer $k$, define that

$$\varphi_k(\Lambda) = 0,$$

$$\varphi_k(a_jx) = j + k\varphi_k(x), \quad j \in \{1, \ldots, k\}, \ x \in W(\{a_1, \ldots, a_k\}).$$

Obviously, $\varphi_k$ is a one-to-one mapping from $W(\{a_1, \ldots, a_k\})$ onto $N$. We denote the inverse mapping of $\varphi_k$ by $\varphi_k^{-1}$, and refer to $\varphi_k^{-1}(y)$ as the *k-ary code* of $y$, for $y$ in $N$. Also, we refer to $\varphi_k^{-1}(\varphi_h(x))$ as the *k-ary code* of $x$, for $x$ in $W(\{a_1, \ldots, a_k\})$.

Let $M$ be an automaton, $f$ and $g$ two $n$-ary functions over $N$, and $n < m$. Denote $|\mathscr{A}_j| = p_j$, $j = 1, \ldots, m$.

**Definition.** We say that *$f$ is computable in time $g$ by $M$*, if there exists an internal state $x_0$ of $M$ such that for any $x_1, \ldots, x_n$ in $N$, when $f(x_1, \ldots, x_n)$ is defined $\xi_M^t(\gamma)$ is a halting state and its $(n+2)$th component is $\varphi_{p_{n+1}}^{-1}(f(x_1, \ldots, x_n))$, and when $f(x_1, \ldots, x_n)$ is undefined either $t$ is undefined or $\xi_M^t(\gamma)$ is not a halting state, where $t$ denotes $g(x_1, \ldots, x_n)$, and $\gamma = (x_0, \varphi_{p_1}^{-1}(x_1), \ldots, \varphi_{p_n}^{-1}(x_n), \Lambda, \ldots, \Lambda)$.

We say that a state $\gamma$ of $M$ is *in a loop* if there exist $r > 1$ and $r$ distinct states $\gamma_1, \ldots, \gamma_r$ such that $\xi_M(\gamma_j) = \gamma_{j+1}$, $j = 1, \ldots, r-1$, $\xi_M(\gamma_r) = \gamma_1 = \gamma$, and $\eta_i(\gamma_{j0}) = 0$, $\lambda_i(\gamma_{j0}) = \Lambda$, $i = 1, \ldots, m$, $j = 1, \ldots, r$, $\gamma_{j0}$ being the scan state of $\gamma_j$. We say that *$f$ is strongly computable in time $g$ by $M$*, if there exists an internal state $x_0$ of $M$ such that for any $x_1, \ldots, x_n$ in $N$, when $f(x_1, \ldots, x_n)$ is defined, $\xi_M^t(\gamma)$ is a halting state of the form

$$(x_0', \varphi_{p_1}^{-1}(x_1), \ldots, \varphi_{p_n}^{-1}(x_n), \varphi_{p_{n+1}}^{-1}(f(x, \ldots, x)), x_{n+2}', \ldots, x_m'),$$

and when $f(x_1, \ldots, x_n)$ is undefined either $t$ is undefined or $\xi_M^t(r)$ is in a loop, where $t$ denotes $g(x_1, \ldots, x_n)$, and $\gamma = (x_0, \varphi_{p_1}^{-1}(x_1), \ldots, \varphi_{p_n}^{-1}(x_n), \Lambda, \ldots, \Lambda)$.

We say that $f$ *is computable in space* $g$ *by* $M$, if there exists an internal state $x_0$ of $M$ such that for any $x_1, \ldots, x_n$ in $N$, $f(x_1, \ldots, x_n)$ is defined if and only if $\mathrm{lm}_M\,\gamma$ is defined and for any $r$ in $N$ the length of each external state in $\xi^r_M(\gamma)$ is at most $g(x_1, \ldots, x_n)$, and in this case the $(n+2)$th component of $\mathrm{lm}_M\,\xi$ is $\varphi^{-1}_{p_{n+1}}(f(x_1, \ldots, x_n))$, where $\gamma = (x_0, \varphi^{-1}_{p_1}(x_1), \ldots, \varphi^{-1}_{p_n}(x_n), \Lambda, \ldots, \Lambda)$.

We say that $f$ *is strongly computable in space* $g$ *by* $M$, if there exists an internal state $x_0$ of $M$ such that for any $x_1, \ldots, x_n$, $r$ in $N$, the length of each external state in $\xi^r_M(\gamma)$ is at most $g(x_1, \ldots, x_n)$ if $g(x_1, \ldots, x_n)$ is defined, and $f(x_1, \ldots, x_n))$ is defined if and only if $\mathrm{im}_M\,\gamma$ and $g(x_1, \ldots, x_n)$ are defined and in this case $\mathrm{lm}_M\,\gamma$ is of the form

$$(x'_0, \varphi^{-1}_{p_1}(x_1), \ldots, \varphi^{-1}_{p_n}(x_n), \varphi^{-1}_{p_{n+1}}(f(x_1, \ldots, x_n)), x'_{n+2}, \ldots, x'_m),$$

where $\gamma = (x_0, \varphi^{-1}_{p_1}(x_1), \ldots, \varphi^{-1}_{p_n}(x_n), \Lambda, \ldots, \Lambda)$.

Obviously, if $f$ is strongly computable in time (or space) $g$ by $M$, then $f$ is computable in time (or space) $g$ by $M$, and $f$ is strongly computable in time (or space) $h$ by $M$ when $g$ and $h$ are total functions and $g \leqslant h$.

**Lemma 1.** *Let $f$, $g$ and $h$ be $n$-ary total functions. If $g \leqslant h$ and $f$ is computable in time (or space) $g$ by $M$, then $f$ is computable in time (or space) $h$ by $M$.*

For any function $g$ over $N$, we define time (or space) complexity classes

$\mathrm{TC}(g)$ (or $\mathrm{SC}(g)$)

$= \{f \mid$ there exists an automaton $M$ such that $f$ is computable in time (or space) $g$ by $M\}$,

$\mathrm{TC}^1(g)$ (or $\mathrm{SC}^1(g)$)

$= \{f \mid$ there exists an unary automaton $M$ such that $f$ is computable in time (or space) $g$ by $M\}$.

In [7], Ackermann's functions are defined as $f_0(x, y) = x+1$, $f_1(x, y) = x+y$, $f_2(x, y) = xy$, $f_{n+1}(x, 0) = 1$, $f_{n+1}(x, y+1) = f_n(x, f_{n+1}(x, y))$, $n \geqslant 2$. For $i \geqslant 1$, $n \geqslant 0$ and $k \geqslant 1$, we denote $f^{(n)}_{i,k}(x_1, \ldots, x_n) = f_i(\max(x_1, \ldots, x_n, 2), k)$. For $i \geqslant 1$, define that

$$T_i = \bigcup_{\substack{n \geqslant 0\\ k \geqslant 1}} \mathrm{TC}(f^{(n)}_{i,k}), \qquad T^1_i = \bigcup_{\substack{n \geqslant 0\\ k \geqslant 1}} \mathrm{TC}^1(f^{(n)}_{i,k}),$$

$$S_i = \bigcup_{\substack{n \geqslant 0\\ k \geqslant 1}} \mathrm{SC}(f^{(n)}_{i,k}), \qquad S^1_i = \bigcup_{\substack{n \geqslant 0\\ k \geqslant 1}} \mathrm{SC}^1(f^{(n)}_{i,k}).$$

Denote the set of all total functions over $N$ by TOL. For $i \geqslant 1$, define that

$$T'_i = T_i \cap \mathrm{TOL}, \qquad T^{1\prime}_i = T^1_i \cap \mathrm{TOL},$$

$$S'_i = S_i \cap \mathrm{TOL}, \qquad S^{1\prime}_i = S^1_i \cap \mathrm{TOL}.$$

From [7, 9], we have following properties:

(1) $f_n(x, y+1) > f_n(x, y) > y$ for $n \geqslant 1, x \geqslant 2, y \geqslant 0$;

(2) $f_n(x+1, y) \geqslant f_n(x, y)$ for $n \geqslant 0, x \geqslant 0, y \geqslant 0$;

(3) $f_n(x, 1) = x$ for $n \geqslant 2, x \geqslant 0$;

(4) $f_n(1, y) > y$ for $n \geqslant 3, x \geqslant 2, y \geqslant 0$;

(5) $f_{n+1}(x, y) \geqslant f_n(x, y)$ for $n \geqslant 2, x \geqslant 2, y \geqslant 0$;

(6) $f_n(f_n(x, y), z) \leqslant f_n(x, yz)$ for $n \geqslant 2, x \geqslant 2, y \geqslant 0, z \geqslant 0$;

(7) $f_{n-1}(f_n(x, y), f_n(x, z)) \leqslant f_n(x, y+z)$ for $n \geqslant 2, x \geqslant 2, y \geqslant 0, z \geqslant 0$;

(8) $f_{m_0-1}(f_{m_1}(x, y), f_{m_2}(x, z)) \leqslant f_n(x, y+z)$ for $n \geqslant m_i \geqslant 2, i = 0, 1, 2, x \geqslant 2, y \geqslant 1, z \geqslant 0$;

(9) $f_{m_1}(x, y_1) + \cdots + f_{m_r}(x, y_r) \leqslant f_n(x, y_1 + \cdots + y_r)$ for $n \geqslant m_i \geqslant 2, y_i \geqslant 1$ (or $2 \leqslant n \geqslant m_i \geqslant 1, y_i \geqslant 2$), $i = 1, \ldots, r, x \geqslant 2$;

(10) $f_n(f_m(x, y), z) \leqslant f_{\max(n,m)}(x, yz)$ for $n \geqslant 2, m \geqslant 2, x \geqslant 2, y \geqslant 1, z \geqslant 0$.

For brevity, when constructing an automaton we use the following notations:

- $i \Rightarrow j$: Clean the $j$th tape, then send the $i$th external state to it, the $i$th tape vanishes;
- $i \Rightarrow i \,\&\, j$: Similar to $i \Rightarrow j$ but with the $i$th external state unchanged;
- $x \Rightarrow j$: Clean the $j$th tape, then send the word $x$ to it;
- $k - \mathrm{C}i \Rightarrow j$: Clean the $j$th tape, then send the $k$-ary code of the $i$th external state to it, with the $i$th external state unchanged;
- $f(x_1, \ldots, x_n) \Rightarrow j$: Clean the $j$th tape, then send (the $k$-ary code of) $f(x_1, \ldots, x_n)$ to it ($k$ being the number of symbols in the $j$th external alphabet);
- $\mathrm{MAX}_1(i_1, i_2) \Rightarrow i_1$: Clean the $i_1$th tape, then send the $p_1$-ary code of $\max(x_1, x_2)$ to it, the $i_2$th tape vanishes, where $p_j$-ary code of $x_j$ is the $i_j$th external state, and $p_j$ is the number of symbols in the $i_j$th external alphabet, $j = 1, 2$;
- $\mathrm{MAX}_2(i_1, i_2) \Rightarrow i_1$: Similar to $\mathrm{MAX}_1(i_1, i_2) \Rightarrow i_1$ but with the $i_2$th external state unchanged;
- $\mathrm{E}i$: Erase the $i$th external scan state;
- $\mathrm{W}i \Rightarrow j$: Write the $i$th external state on the $j$th tape, the $i$th tape vanishes;
- $\mathrm{W}x \Rightarrow j$: Write the word $x$ on the $j$th tape.

We denote the $i$th external state by $\langle i \rangle$, the length of a word $x$ by $|x|$, and $x^0 = \Lambda$, $x^{n+1} = xx^n$, $n \geqslant 0$.

**Lemma 2.** *Let* $p \geqslant 1$ *and* $\psi_p(x) = \sum_{i=1}^{x} p^{i-1}$. *Then for any* $r \geqslant 1$, *there exists an automaton* $M$ *and* $k \geqslant 1$ *such that* $|\mathscr{A}_1| = r$, $l_j = r_j = 1$, $j = 1, 2$, *and* $\psi_p$ *is strongly computable in time* $f_{2,k}^{(1)}$ *by* $M$.

**Proof.** In case of $r = 1$, we construct an automaton $M$ with $|\mathscr{A}_1| = |\mathscr{A}_3| = 1$, $|\mathscr{A}_2| = p$, and $l_i = r_i = 1$, $i = 1, 2, 3$. $M$ behaves as follows:

(1) $1 \Rightarrow 3$.

(2) $3 \Rightarrow 1$, $a_1^l \Rightarrow 2$, $l$ being $|\langle 3 \rangle|$.

It is easy to see that $M$ computes $\psi_p(x)$ and the number of steps is $(x+1) + (x+1) \leqslant 3\max(x, 2) = f_{2,3}^{(1)}(x)$. From Lemma 1, taking $k = 3$, Lemma 2 follows

In case of $r > 1$, we construct an automaton $M$ with $|\mathscr{A}_1| = |\mathscr{A}_5| = r$, $|\mathscr{A}_2| = p$, $|\mathscr{A}_3| = |\mathscr{A}_4| = 1$, $l_1 = \cdots = l_5 = 1$, $r_1 = r_2 = r_5 = 1$, and $r_3 = r_4 = r$. $M$ behaves as follows:

(1) $1 \Rightarrow 5$, $a_1 \Rightarrow 3$.

(2) If the 5th external scan state is a symbol, say $a_i$, then E5, W$a_i \Rightarrow 1$, $\langle 3 \rangle^i \Rightarrow 2$, $\langle 3 \rangle^r \Rightarrow 4$, $\Lambda \Rightarrow 3$, and go to (3); otherwise halt.

(3) If the 5th external scan state is a symbol, say $a_i$, then E5, W$a_i \Rightarrow 1$, $\langle 4 \rangle^i \Rightarrow 2$, $\langle 4 \rangle^r \Rightarrow 3$, $\Lambda \Rightarrow 4$, and go to (2); otherwise halt.

It is easy to see that $M$ computes $\psi_p(x)$ and the number of steps is at most

$$(|x|+1) + \left(|x| + x + \sum_{i=1}^{|x|} r^{i-1}\right) + 1 \leqslant 4\max(x, 2) = f_{2,4}^{(1)}(x).$$

From Lemma 1, taking $k = 4$, Lemma 2 follows.

Note that in Lemma 2, when $p = 1$, the automaton $M$ transforms the $r$-ary codes of the non-negative integers to their unary codes.

**Lemma 3.** *For any* $p \geqslant 1$, *there exists an automaton* $M$ *and* $k \geqslant 1$ *such that* $|\mathscr{A}_2| = p$, $l_j = r_j = 1$, $j = 1, 2$, *and* $|x|$ *is strongly computable in time* $f_{2,k}^{(1)}$ *by* $M$, *where* $|x|$ *denotes* $|\varphi_h^{-1}(x)|$, $h = |\mathscr{A}_1|$.

**Proof.** In case of $p = 1$, we construct an automaton $M$ which behaves as follows:

(1) $1 \Rightarrow 3$.

(2) $3 \Rightarrow 1$, $a_1^l \Rightarrow 2$, $l$ being $|\langle 3 \rangle|$.

Clearly, $M$ computes $|x|$ and the number of steps is $(|x|+1) + (|x|+1) \leqslant 3\max(x, 2)$. From Lemma 1, taking $k = 3$, Lemma 3 follows.

In case of $p > 1$, we construct an automaton $M$ with $l_1 = l_2 = l_5 = 1$, $l_3 = l_4 = p$, and $r_1 = \cdots = r_5 = 1$. $M$ behaves as follows:

(1) $1 \Rightarrow 5$.

(2) $5 \Rightarrow 1$, $a_1^l \Rightarrow 3$, $l$ being $|\langle 5 \rangle|$.

(3) If $|\langle 3 \rangle| \geqslant p$, then go to (4); if $0 < |\langle 3 \rangle| < p$, then go to (5); otherwise halt.

(4) Let the 3rd external scan state be $a_1^i$. If $i = p$, then E3, W$a_1 \Rightarrow 4$, and repeat (4); otherwise, E3, W$a \Rightarrow 2$, set BS as $s$, and go to (6), where BS represents the

borrow state contained in the internal state, and

$$(a,s)=\begin{cases}(a_i,0) & \text{if } i>0 \text{ and BS is } 0,\\ (a_{i-1},0) & \text{if } i>1 \text{ and BS is } 1,\\ (a_p,1) & \text{if } i=1 \text{ and BS is } 1,\\ (a_p,1) & \text{if } i=0 \text{ and BS is } 0,\\ (a_{p-1},1) & \text{if } i=0 \text{ and BS is } 1.\end{cases}$$

(5) Let the 3rd external scan state be $a_1^i$. E3, W$a_{i-1}\Rightarrow 2$ (if $i>1$ and BS is 1) or W$a_i\Rightarrow 2$ (if BS is 0), and halt.

(6) If $|\langle 4\rangle|\geq p$, then go to (7); otherwise, go to (8).

(7) Let the 4th external scan state be $a_1^i$. If $i=p$, then E4, W$a_1\Rightarrow 3$, and repeat (7); otherwise, E4, W$a\Rightarrow 2$, set BS as $s$, and go to (3), where $a$ and $s$ are defined as in (4).

(8) Let the 4th external scan state be $a_1^i$. E4, W$a_{i-1}\Rightarrow 2$ (if $i>1$ and BS is 1) or W$a_i\Rightarrow 2$ (if BS is 0), and halt.

It is easy to see that $M$ computes $|x|$ and the number of steps is at most[1]

$$\begin{aligned}&2(|x|+1)+q(|x|,p)+2+q(q(|x|,p),p)+2+\cdots+1\leq\\ &\leq 2(|x|+1)+\frac{|x|}{p}+\frac{|x|}{p^2}+\cdots+2(\log_p|x|+1)+1\\ &\leq 4|x|+5+\frac{|x|}{p-1}\leq 5x+5\leq 8\max(x,2).\end{aligned}$$

From Lemma 1, taking $k=8$, Lemma 3 follows.

Note that in Lemma 3, when $|\mathscr{A}_1|=1$, the automaton $M$ transforms the unary codes of the non-negative integers to their $p$-ary codes.

**Lemma 4.** *For any $i$, $r$, $s$, $t\geq 1$, there exists an automaton $M$ and $k\geq 1$ such that $|\mathscr{A}_1|=r$, $|\mathscr{A}_2|=s$, $|\mathscr{A}_3|=t$, $l_j=r_j=1$, $j=1,2,3$, and $f_i$ is strongly computable in time $f^{(2)}_{i+1,k}$ by $M$.*

**Proof.** By induction on $i$.

*Basis.* $i=1$. $f_1(x,y)=x+y$. We construct an automaton $M$ which behaves as follows:

(1) $1-\mathrm{C}1\Rightarrow 4$.

(2) $1-\mathrm{C}2\Rightarrow 5$.

(3) $\mathrm{W}5\Rightarrow 4$.

(4) $t-\mathrm{C}4\Rightarrow 3$.

[1] When $x$ is divided by $y$, the quotient and the remainder are denoted by $q(x,y)$ and $r(x,y)$ respectively.

It is easy to see that $M$ computes $f_2(x_1, x_2)$ with the 1st and 2nd external states unchanged. Using Lemmas 2 and 3, we have that the number of steps is at most

$$4\max(x_1,2)+4\max(x_2,2)+(x_2+1)+\max(x_1+x_2,2)\leqslant 26\max(x_1,x_2,2).$$

From Lemma 1, taking $k=26$, Lemma 4 follows.

*Induction step.* $i>1$. Assume that for any $r', s', t'\geqslant 1$, there exists an automaton $M'$ and $k'\geqslant 1$ such that $|\mathscr{A}'_1|=r'$, $|\mathscr{A}'_2|=s'$, $|\mathscr{A}'_3|=t'$, $l'_j=r'_j=1$, $j=1,2,3$, and $f_{i-1}$ is strongly computable in time $f^{(2)}_{i,k'}$ by $M'$. Given $r,s,t\geqslant 1$, we construct an automaton $M$ which behaves as follows:

(1) $1\Rightarrow 1\ \&\ 4$.

(2) $1-\mathrm{C}2\Rightarrow 7$.

(3) In case of $i\geqslant 3$, $a_1\Rightarrow 6$.

(4) If $\langle 7\rangle=\Lambda$, then $t-\mathrm{C}6\Rightarrow 3$ and halt; otherwise, E7, $6\Rightarrow 5$, and go to (5).

(5) Simulate $M'$ ($r'=r$, $s'=t'=|\mathscr{A}_5|=|\mathscr{A}_6|$) with the 4th, 5th and 6th tapes of $M$ as the 1st, 2nd and 3rd tapes of $M'$ respectively, go to (4) as soon as $M'$ reaches a halting state.

By the induction hypothesis and Lemmas 2 and 3, it is easy to show that $M$ computes $f_i(x_1, x_2)$ with the 1st and 2nd external state unchanged and the number of steps is at most

$$(2|x_1|+2)+4\max(x_2,2)+1+\sum_{j=0}^{x_2-1}[f_i(\max(x_1,f_i(x_1,j),2),k')+1]$$

$$+\sum_{j=0}^{x_2-1}[f_i(x_1,j)+2]+8\max(f_i(x_1,x_2),2)+1.$$

Denote $w=\max(x_1,x_2,2)$. Since $f_i(x_1,j)\leqslant f_i(w,j)\leqslant f_i(w,w)=f_i(f_{i+1}(w,1), f_{i+1}(w,1))\leqslant f_{i+1}(w,2)$, we have that the number of steps is at most

$$\begin{aligned}
&2w+2+4w+1+x_2f_i(f_{i+1}(w,2),k')+x_2+x_2f_{i+1}(w,2)+2x_2+8f_{i+1}(w,2)+1\leqslant\\
&\quad\leqslant 9w+4+wf_{i+1}(w,2k')+(w+8)f_{i+1}(w,2)\\
&\quad\leqslant 9w+4+f_2(f_{i+1}(w,2k'),f_{i+1}(w,1))+f_2(f_{i+1}(w,2),f_{i+1}(w,4))\\
&\quad\leqslant f_3(w,5)+f_{i+1}(w,2k'+1)+f_{i+1}(w,6)\\
&\quad\leqslant f_{i+1}(w,2k'+12).
\end{aligned}$$

Taking $k=2k'+12$, it follows that $f_i$ is strongly computable in time $f^{(2)}_{i+1,k}$ by $M$.

**Corollary 1.** *For any $i\geqslant 1$, there exists an unary automaton $M$ and $k\geqslant 1$ such that $l_j=r_j=1$, $j=1,2,3$, and $f_i$ is strongly computable in time $f^{(2)}_{i+1,k}$ by $M$.*

**Corollary 2.** $f_i\in T^{\mathrm{f}}_{i+1}$, $f_i\in T^{1\mathrm{f}}_{i+1}$, $i\geqslant 1$.

Using Lemma 4, it is not difficult to show the following result.

**Lemma 5.** *For any $i, r \geqslant 1$ and $c \geqslant 0$, there exists an automaton $M$ and $k \geqslant 1$ such that $|\mathscr{A}_1| = |\mathscr{A}_2| = r$, $l_1 = l_2 = r_1 = r_2 = 1$ in case of* $\max(i, r) \geqslant 2$, *and $f_{i,c}$ is strongly computable in time $f_{i,k}^{(1)}$ by $M$, where $f_{i,c}(x) = f_i(x, c)$ for $x$ in $N$.*

**Corollary 1.** *For any $i \geqslant 1$ and $c \geqslant 0$, there exists an unary automaton $M$ and $k \geqslant 1$ such that $l_1 = l_2 = r_1 = r_2 = 1$ in case of $i \geqslant 2$, and $f_{i,c}$ is strongly computable in time $f_{i,k}^{(1)}$ by $M$.*

**Corollary 2.** *For any $i \geqslant 1$ and $c \geqslant 0$, we have that $f_{i,c} \in T_i^1$ and $f_{i,c} \in T_i^{1\prime}$.*

It is easy to show the following useful fact: if $f$ is computable in time $g$ by $M$, then there exists an automaton $M'$ such that $r_j' = 1$, $j = 1, \ldots, m'$, and $f$ is computable in time $rg$ by $M'$, where $r = \max(r_1, \ldots, r_m, 1)$.

**Lemma 6.** *Let $i \geqslant 2$. If an $n$-ary function $f \in T_i$, then for any $p_1, \ldots, p_n \geqslant 1$, there exists an automaton $M$ and $k \geqslant 1$ such that $|\mathscr{A}_j| = p_j$, $l_j = r_j = 1$, $j = 1, \ldots, n$, and $f$ is strongly computable in time $f_{i,k}^{(n)}$ by $M$.*

**Proof.** From $f \in T_i$, there exists an automaton $M'$ and $b, c \geqslant 1$ such that $f$ is computable in time $bf_{i,c}^{(n)}$ and $r_1' = \cdots = r_{m'}' = 1$. Denote $|\mathscr{A}_j'| = h_j$, $j = 1, \ldots, m'$. Given $p_1, \ldots, p_n \geqslant 1$, we construct an automaton $M$ with $|\mathscr{A}_j| = p_j$, $l_j = r_j = 1$, $j = 1, \ldots, n$, and $\mathscr{A}_{n+1} = \mathscr{A}_{n+1}'$, $l_{n+1} = l_{n+1}'$, $r_{n+1} = r_{n+1}'$, and $\mathscr{A}_{n+1+j} = \mathscr{A}_j'$, $l_{n+1+j} = l_j'$, $r_{n+1+j} = r_j'$, $j = 1, \ldots, n$, $n+2, \ldots, m'$, and $|\mathscr{A}_j| = l_j = r_j = 1$, $j = n+m'+2, \ldots, n+m'+4$, and $\mathscr{A}_{n+m'+4+j} = \mathscr{A}_j'$, $l_{n+m'+4+j} = 1$, $r_{n+m'+4+j} = 1$, $j = 1, \ldots, n$. $M$ behaves as follows:

(1) $a_1^2 \Rightarrow n+m'+2$.

$(1+j)$ $(1 \leqslant j \leqslant n)$

(i) $1 - Cj \Rightarrow n+m'+3$.

(ii) $h_j - Cn+m'+3 \Rightarrow n+m'+4+j$.

(iii) $n+m'+4+j \Rightarrow n+1+j$.

(iv) $\mathrm{MAX}_1(n+m'+2, n+m'+3) \Rightarrow n+m'+2$.

$(2+n)$. (i) $f_i(\varphi_1(\langle n+m'+2\rangle), c) \Rightarrow n+m'+3$. (Using Lemma 5, the number of steps is at most $f_{i,k'}^{(1)}(\varphi_1(\langle n+m'+2\rangle))$ for some $k' \geqslant 1$.)

(ii) $\varphi_1^{-1}(b\varphi_1(\langle n+m'+3\rangle)) \Rightarrow n+m'+4$.

$(3+n)$ If $\langle n+m'+4\rangle = \Lambda$, then go to $(5+n)$, otherwise E$n+m'+4$ and go to $(4+n)$.

$(4+n)$ Simulate one step of $M'$ with the $(n+1+1)$th, $\ldots$, $(n+1+n)$th, $(n+1)$th, $(n+1+n+2)$th, $\ldots$, $(n+1+m')$th tapes of $M$ as the 1st, $\ldots$ $m'$th tapes of $M'$ respectively, and go to $(3+n)$.

$(5+n)$ If $M'$ reaches a halting state, then halt, otherwise enter in a loop.

It is easy to see that $M$ computes $f(x_1,\dots,x_n)$ with the 1st, …, $n$th tapes unchanged and when $f(x_1,\dots,x_n)$ is defined the number of steps is at most

$$2+\sum_{j=1}^{n}[4\max(x_j,2)+8\max(x_j,2)+(|x_j|+1)+2(\max(2,x_1,\dots,x_j)+1)]$$
$$+f_i(\max(x_1,\dots,x_n,2),k')+(bf_i(\max(x_1,\dots,x_n,2),c)+1)$$
$$+2bf_i(\max(x_1,\dots,x_n,2),c)+2.$$

Denote $w=\max(x_1,\dots,x_n,2)$. Then the number of steps is at most

$$15nw+3n+5+f_i(w,k')+3bf_i(w,c)\leqslant$$
$$\leqslant f_2(w,17n+3)+f_i(w,k')+f_i(w,3bc)$$
$$\leqslant f_i(w,17n+3+k'+3bc).$$

Taking $k=17n+3+k'+3bc$, Lemma 6 follows.

**Corollary 1.** *Let $i\geqslant 2$. If an $n$-ary function $f\in T_i^1$, then there exists an unary automaton $M$ and $k\geqslant 1$ such that $l_j=r_j=1$, $j=1,\dots,n$, and $f$ is strongly computable in time $f_{i,k}^{(n)}$ by $M$.*

**Corollary 2.** *If an $n$-ary function $f\in T_1$, then for any $p_1,\dots,p_n\geqslant 1$, there exists an automaton $M$ and $k\geqslant 1$ such that $|\mathcal{A}_j|=p_j$, $l_j=r_j=1$, $j=1,\dots,n$, and $f$ is strongly computable in time $f_{2,k}^{(n)}$ by $M$.*

**Corollary 3.** *If an $n$-ary function $f\in T_1^1$, then there exists an unary automaton $M$ and $k\geqslant 1$ such that $l_j=r_j=1$, $j=1,\dots,n$, and $f$ is strongly computable in time $f_{2,k}^{(n)}$ by $M$.*

**Lemma 7.** *Let $g$ and $h$ be two $n$-ary total functions and $g\leqslant h$. If $f$ is strongly computable in time* (or space) *$g$ by $M$, then $f$ is strongly computable in time* (or space) *$h$ by $M$.*

**Theorem 1.** $T_i\subseteq T_{i+1}$, $i\geqslant 1$.

**Proof.** Let $f$ be an $n$-ary function in $T_i$. Using Lemma 6 and Corollary 2, there exists an automaton $M$ and $k\geqslant 1$ such that $f$ is strongly computable in time $f_{\max(2,i),k}^{(n)}$ by $M$. Since $f_{\max(2,i)}(\max(x_1,\dots,x_n,2),k)\leqslant f_{i+1}(\max(x_1,\dots,x_n,2),k)$, using Lemma 7, $f$ is strongly computable in time $f_{i+1,k}^{(n)}$ by $M$. It follows that $f\in T_{i+1}$.

**Corollary 1.** $T_i^t\subseteq T_{i+1}^t$, $i\geqslant 1$.

**Corollary 2.** $T_i^1\subseteq T_{i+1}^1$, $i\geqslant 1$.

**Corollary 3.** $T_i^{1t}\subseteq T_{i+1}^{1t}$, $i\geqslant 1$.

**Lemma 8.** $p^{x^2} \in T_3^t - T_2^t, p > 1$.

**Proof.** We construct an automaton $M$ with $|\mathscr{A}_j| = 1$, $j = 1, 3, 4, 5$, $|\mathscr{A}_2| = p$ and $l_j = r_j = 1$, $j = 1, \ldots, 5$. $M$ behaves as follows:

(1) If $\langle 1 \rangle = \Lambda$, then $a_1 \Rightarrow 2$ and halt; otherwise $1 \Rightarrow 3$, $1 \Rightarrow 5$ and go to (2).

(2) E3, $5 \Rightarrow 1$, W5 $\Rightarrow 4$.

(3) If $\langle 3 \rangle = \Lambda$, then go to (4), otherwise $1 \Rightarrow 5$ and go to (2).

(4) $a_p a_{p-1}^{l-1} \Rightarrow 2$, $l$ being $|\langle 4 \rangle|$.

It is easy to see that $M$ computes $p^{x^2}$ and the number of steps is at most

$$(x+1)+(x+1)x+(x+1)(x-1)+1+(x^2+1) \leqslant$$
$$\leqslant 5[\max(x,2)]^2 \leqslant f_3(\max(x,2),5).$$

From Lemma 1, $p^{x^2}$ is computable in time $f_{3,5}^{(1)}$ by $M$. Therefore, $p^{x^2} \in T_3^t$.

Let $f$ be an $n$-ary function in $T_2^t$. Then there exists an automaton $M$ and $k \geqslant 1$ such that $f$ is computable in time $f_{2,k}^{(n)}$ by $M$. Denote $w = \max(x_1, \ldots, x_n, 2)$ and $|\mathscr{A}_{n+1}| = s$. In case of $s > 1$, we have that

$$f(x_1, \ldots, x_n) \leqslant 2s^{r_{n+1} f_2(w,k)} \leqslant (2s^{r_{n+1}k})^w.$$

In case of $s = 1$, we have that

$$f(x_1, \ldots, x_n) \leqslant r_{n+1} f_2(w,k) = (r_{n+1}k)w.$$

Since $\lim_{w\to\infty} p^{w^2}/c^w = \infty$ and $\lim_{w\to\infty} p^{w^2}/cw = \infty$ for $c > 0$, we conclude that $p^{x^2} \notin T_2^t$.

**Corollary.** (a) *For any $n$-ary function $f$ in $T_2^t$, there exists $c > 1$ such that $f(x_1, \ldots, x_n) \leqslant c^{\max(x_1,\ldots,x_n,2)}$.*

(b) *For any $n$-ary function $f$ in $T_2^{1t}$, there exists $c \geqslant 1$ such that $f(x_1, \ldots, x_n) \leqslant c \max(x_1, \ldots, x_n, 2)$.*

Analogously, we can show the following result.

**Lemma 9.** $p^{x_1^{x_2}} \in T_4^t - T_3^t, p > 1$.

**Corollary.** (a) *For any $n$-ary function $f$ in $T_3^t$, there exist $c > 1$ and $k \geqslant 1$ such that $f(x_1, \ldots, x_n) \leqslant c^{[\max(x_1,\ldots,x_n,2)]^k}$.*

(b) *For any $n$-ary function $f$ in $T_3^{1t}$, there exists $k \geqslant 1$ such that $f(x_1, \ldots, x_n) \leqslant [\max(x_1, \ldots, x_n, 2)]^k$.*

**Lemma 10.** (a) *If an $n$-ary function $f \in T_i$, $i \geqslant 4$, then there exists $k \geqslant 1$ such that*

$$f(x_1, \ldots, x_n) \leqslant f_i(\max(x_1, \ldots, x_n, 2), k)$$

*(when $f(x_1, \ldots, x_n)$ is defined).*

(b) *If an n-ary function* $f \in T_i^1$, $i \geq 2$, *then there exists* $k \geq 1$ *such that*

$$f(x_1, \ldots, x_n) \leq f_i(\max(x_1, \ldots, x_n, 2), k)$$

(*when* $f(x_1, \ldots, x_n)$ *is defined*).

**Proof.** (a) Let $f$ be an $n$-ary function in $T_i$, $i \geq 4$. Then there exists an automaton $M$ and $c \geq 1$ such that $f$ is computable in time $f_{i,c}^{(n)}$ by $M$. Denote $w = \max(x_1, \ldots, x_n, 2)$ and $|\mathscr{A}_{n+1}| = s$. When $f(x_1, \ldots, x_n)$ is defined, the number of steps is at most $f_i(w, c)$. Thus the lengths of the $(n+1)$th external states during the computation are at most $r_{n+1} f_i(w, c)$. It follows that

$$\begin{aligned} f(x, \ldots, x) &\leq 2s^{r_{n+1} f_i(w,c)} \\ &\leq f_3(2s^{r_{n+1}}, f_i(w, c)) \leq f_3(f_2(w, s^{r_{n+1}}), f_i(w, c)) \\ &\leq f_i(w, s^{r_{n+1}} + c). \end{aligned}$$

The proof of (b) is similar to (a).

Note that Lemma 10(a) is not true for $1 \leq i \leq 3$. In fact, $p^x \in T_i$ for $1 \leq i \leq 3$ and $p \geq 2$, but there is no $c \geq 1$ such that $p^x \leq x + c$ or $p^x \leq cx$ or $p^x \leq x^c$ for all $x \geq 0$. Also, Lemma 10(b) is not true for $i = 1$. In fact, $2x \in T_1^1$ but there is no $c \geq 1$ such that $2x \leq x + c$ for all $x \geq 0$.

**Lemma 11.** $f_i \in T_{i+1}^t - T_i^t$, $i \geq 4$.

**Proof.** Suppose that $f_i \in T_i^t$. From Lemma 10, there exists $k \geq 1$ such that $f_i(x, y) \leq f_i(\max(x, y, 2)), k)$. Taking $x = y = k + 1$, we obtain $f_i(k+1, k+1) \leq f_i(k+1, k)$. This contradicts to $f_i(k+1, k) < f_i(k+1, k+1)$. Thus $f_i \notin T_i^t$. From Corollary 2 of Lemma 4, we have that $f_i \in T_{i+1}^t$. Lemma 11 follows.

From Theorem 1 and its Corollary 1, and Lemmas 8, 9 and 11, we have

**Theorem 2.** $T_i \subset T_{i+1}$, $i \geq 2$.

**Corollary.** $T_i^t \subset T_{i+1}^t$, $i \geq 2$.

**Lemma 12.** $f_i \in T_{i+1}^{1t} - T_i^{1t}$, $i \geq 2$.

**Proof.** Similar to Lemma 11.

From Lemma 12, Corollaries 2 and 3 of Theorem 1, we have

**Theorem 3.** $T_i^1 \subset T_{i+1}^1$, $i \geq 2$.

**Corollary.** $T_i^{1t} \subset T_{i+1}^{1t}$, $i \geq 2$.

**Theorem 4.** $T_1 = T_2$.

**Proof.** From Theorem 1, we have that $T_1 \subseteq T_2$. To prove $T_2 \subseteq T_1$, let $f$ be an $n$-ary function in $T_2$. From Lemma 6, there exists an automaton $M$ and $k \geq 1$ such that $f$ is strongly computable in time $f_{2,k}^{(n)}$ by $M$. Without loss of generality, we assume that $l_{n+1} > 0$. Denote $|\mathscr{A}_j| = p_j$, $j = 1, \ldots, m$. We construct an automaton $M^*$ with $m^* = m+1$, and $|\mathscr{A}_j^*| = p_{j'}$, $l_j^* = sl_{j'}$, $r_j^* = sr_{j'}$, $j = 1, \ldots, n, n+2, \ldots, m+1$, and $|\mathscr{A}_{n+1}^*| = \sum_{i=1}^{sl_{n+1}} p_{n+1}^i$ (if $p_{n+1} > 1$) or 1 (if $p_{n+1} = 1$), $l_{n+1}^* = l_{m+1}^* - 1$, $r_{n+1}^* = 1$ (if $p_{n+1} > 1$) or $l_{m+1}^*$ (if $p_{n+1} = 1$), where $s$ is some integer $\geq (1 + r_{n+1}/l_{n+1})k$, and $j' = j$ (if $1 \leq j \leq n$ or $n+2 \leq j \leq m$) or $n+1$ (if $j = m+1$). In the interior of $M^*$, there are $m$ input-registers, say $\mathrm{IR}_j$, $j = 1, \ldots, n, n+2, \ldots, m+1$, and $m$ output-registers, say $\mathrm{OR}_j$, $j = 1, \ldots, n, n+2, \ldots, m+1$. A left segment $x_j'$ of the $j'$th external state $x_{j'}$ of $M$ with $\min(|x_{j'}|, sl_{j'}) \leq |x_j'| < 2sl_{j'}$ is stored in $\mathrm{IR}_j$; a right segment $x_j''$ of $x_{j'}$ with $|x_j''| < 2sr_{j'}$ is stored in $\mathrm{OR}_j$; the remainder segment of $x_{j'}$ is the $j$th external state of $x_j^*$ of $M^*$, that is to say, $x_{j'} = x_j' x_j^* x_j''$ ($j = 1, \ldots, n, n+2, \ldots, m+1$). $M^*$ simulates $s$ steps of $M$ by one step and behaves as follows:

(1) Read the $j$th external scan state into $\mathrm{IR}_j$, $j = 1, \ldots, n, n+2, \ldots, m+1$.

(2) $M^*$ runs one step to simulate $s$ steps of $M$. If $M$ reaches a halting state, then go to (3), otherwise repeat (2).

(3) $\varphi_p^{-1}(\varphi_{p_{n+1}}(x_{m+1}' x_{m+1}^* x_{m+1}'')) \Rightarrow n+1$, where $p$ denotes $|\mathscr{A}_{n+1}^*|$, $x_{m+1}'$ and $x_{m+1}''$ are the contents of $\mathrm{IR}_{m+1}$ and $\mathrm{OR}_{m+1}$ respectively, and $x_{m+1}^*$ is the $(m+1)$th external state of $M^*$.

It is easy to see that $M^*$ computes $f(x_1, \ldots, x_n)$ and when $f(x_1, \ldots, x_n)$ is defined the number of steps is at most[2]

$$1 + \lceil (kw+1)/s \rceil + \lceil r_{n+1} kw / sl_{n+1} \rceil + sl_{n+1} \leq$$

$$\leq \frac{k}{s} w + \frac{kr_{n+1}}{sl_{n+1}} w + sl_{n+1} + 4 \leq w + c,$$

where $w = \max(x_1, \ldots, x_n, 2)$, $c = sl_{n+1} + 4$. It follows that $f$ is computable in time $f_{1,c}^{(n)}$ by $M^*$. We conclude that $f \in T_1$.

**Corollary 1.** $T_1^t = T_2^t$.

**Corollary 2.** $T_1^1 = T_2^1$.

**Corollary 3.** $T_1^{1t} = T_2^{1t}$.

## 3. Computation with space bounded by Ackermann's functions

**Lemma 13.** *For any $r \geq 1$, there exists an automaton $M$ such that $|\mathscr{A}_j| = r$, $l_j = r_j = 1$, $j = 1, \ldots, m$, and $x+1$ is strongly computable in space $|x| + 1$ by $M$, where $|x|$ denotes $|\varphi_r^{-1}(x)|$.*

[2] $\lceil u \rceil$ denotes the least integer $\geq u$.

**Lemma 14.** *For any $r \geqslant 1$, there exists an automaton $M$ such that $|\mathscr{A}_j| = r$, $l_j = r_j = 1$, $j = 1, \ldots, m$, and $x - 1$ is strongly computable in space $|x|$ by $M$, $0 - 1$ being undefined.*

**Lemma 15.** *Let $i \geqslant 2$ and $k \geqslant 1$. If $f$ is strongly computable in time $f_{i,k}^{(n)}$ by $M$, then $f$ is strongly computable in space $f_{i,kr+1}^{(n)}$ by $M$, $r$ being the maximum output-length of $M$.*

**Proof.** Denote $w = \max(x_1, \ldots, x_n, 2)$. Since the tape-lengths increase at most $rc$ in $c$ steps, and $M$ reaches a halting state or enters in a loop in $f_i(w, k)$ steps when computing $f(x_1, \ldots, x_n)$, we have that the tape-lengths are at most

$$\max(|x_1|, \ldots, |x_n|) + rf_i(w, k) \leqslant f_i(w, 1) + f_2(f_i(w, k), r)$$
$$\leqslant f_i(w, 1) + f_i(w, kr) \leqslant f_i(w, kr + 1).$$

**Corollary.** $T_i \subseteq S_i$, $T_i^1 \subseteq S_i^1$, $T_i^t \subseteq S_i^t$, $T_i^{1t} \subseteq S_i^{1t}$, $i \geqslant 2$.

From Lemma 4 and Corollary 1, and Lemma 15, we have

**Lemma 16.** *For any $i, r, s, t \geqslant 1$, there exists an automaton $M$ and $k \geqslant 1$ such that $|\mathscr{A}_1| = r$, $|\mathscr{A}_2| = s$, $|\mathscr{A}_3| = t$, $l_j = r_j = 1$, $j = 1, 2, 3$, and $f_i$ is strongly computable in space $f_{i+1,k}^{(2)}$ by $M$.*

**Corollary 1.** *For any $i \geqslant 1$, there exists an unary automaton $M$ and $k \geqslant 1$ such that $l_j = r_j = 1$, $j = 1, 2, 3$, and $f_i$ is strongly computable in space $f_{i+1,k}^{(2)}$ by $M$.*

**Corollary 2.** $f_i \in S_{i+1}^t$, $f_i \in S_{i+1}^{1t}$, $i \geqslant 1$.

Using Lemma 16 and Corollary 1, we can show the following result.

**Lemma 17.** *For any $i, r, t \geqslant 1$ and $c \geqslant 1$, there exists an automaton $M$ and $k \geqslant 1$ such that $|\mathscr{A}_1| = r$, $|\mathscr{A}_2| = t$, $l_1 = l_2 = r_1 = r_2 = 1$, and $f_{i,c}$ is strongly computable in space $f_{i,k}^{(1)}$ by $M$.*

**Corollary 1.** *For any $i \geqslant 1$ and $c \geqslant 0$, there exists an unary automaton $M$ and $k \geqslant 1$ such that $l_1 = l_2 = r_1 = r_2 = 1$, and $f_{i,c}$ is strongly computable in space $f_{i,k}^{(1)}$ by $M$.*

**Corollary 2.** $f_{i,c} \in S_i^t$, $f_{i,c} \in S_i^{1t}$, $i \geqslant 1$, $c \geqslant 0$.

**Lemma 18.** *Let $i \geqslant 1$. If an $n$-ary function $f \in S_i$, then for any $p_1, \ldots, p_{n+1} \geqslant 1$, there exists an automaton $M$ and $k \geqslant 1$ such that $|\mathscr{A}_j| = p_j$, $l_j = r_j = 1$, $j = 1, \ldots, n + 1$, and $f$ is strongly computable in space $f_{i,k}^{(n)}$ by $M$.*

**Proof.** Since $f \in S_i$, there exists an automaton $M'$ and $c \geqslant 1$ such that $f$ is computable in space $f_{i,c}^{(n)}$ by $M'$. Without loss of generality, we assume that $l'_{n+1} > 0$ and $r'_j = 1$,

$j=1,\dots,m'$. Denote $|\mathscr{A}'_j|=h_j$, $j=1,\dots,m'$. We construct an automaton $M$ with $|\mathscr{A}_j|=p_j$, $l_j=r_j=1$, $j=1,\dots,n+1$, and $|\mathscr{A}_{n+1+j}|=h_j$, $l_{n+1+j}=l'_j$, $r_{n+1+j}=r'_j$, $j=1,\dots,m'$, and $|\mathscr{A}_{n+1+m'+j}|=l_{n+1+m'+j}=r_{n+1+m'+j}=1$, $j=1,\dots,m'+2$. $M$ behaves as follows:

(1) $a_1^2 \Rightarrow n+2+2m'$.

$(1+j)$ $(1\leqslant j\leqslant n)$ $h_j-Cj\Rightarrow n+1+j$, $\mathrm{MAX}_2(n+2+2m', j)\Rightarrow n+2+2m'$.

$(2+n)$ $\varphi_1^{-1}(f_{i,c}(\varphi_1(\langle n+2+2m'\rangle)))\Rightarrow n+3+2m'$.

$(3+n)$ $n+3+2m'\Rightarrow n+1+m'+j$, $j=1,\dots,m'$.

$(3+n+j)$ $(1\leqslant j\leqslant n)$ $\varphi_1^{-1}(\varphi_1(\langle n+1+m'+j\rangle)-|\langle j\rangle|)\Rightarrow n+1+m'+j$.

$(4+2n)$ Simulate one step of $M'$ with the $(n+1+j)$th tape of $M$ as the $j$th tape of $M'$, $j=1,\dots,m'$. Let the increment of the $j$th tape-length of $M'$ in this step be $e_j$, $j=1,\dots,m'$. If there is some $1\leqslant j\leqslant m'$ such that $\varphi_1(\langle n+1+m'+j\rangle)<e_j$, then enter in a loop, otherwise $\varphi_1^{-1}(\varphi_1(\langle n+1+m'+j\rangle)-e_j)\Rightarrow n+1+m'+j$, $j=1,\dots,m'$, and go to $(5+2n)$.

$(5+2n)$ If $M'$ reaches a halting state, then go to $(6+2n)$, otherwise go to $(4+2n)$.

$(6+2n)$ $p_{n+1}-C2n+2\Rightarrow n+1$, and halt.

Using Lemma 17, it is not hard to show that there exists $a\geqslant 1$ such that $M$ computes $f(x_1,\dots,x_n)$ with the 1st, …, $n$th external states unchanged and the tape-lengths are at most $\max(w, f_i(w,a), f_i(w,c))=f_i(w,\max(a,c))$, $w$ being $\max(x_1,\dots,x_n,2)$. Therefore, $f$ is strongly computable in space $f^{(n)}_{i,\max(a,c)}$ by $M$.

**Corollary.** *Let $i\geqslant 1$. If an n-ary function $f\in S_i^1$, then there exists an unary automaton $M$ and $k\geqslant 1$ such that $l_j=r_j=1$, $j=1,\dots,n+1$, and $f$ is strongly computable in space $f^{(n)}_{i,k}$ by M.*

From Lemmas 18 and 7, we have

**Theorem 5.** $S_i\subseteq S_{i+1}$, $i\geqslant 1$.

**Corollary 1.** $S_i^t\subseteq S_{i+1}^t$, $i\geqslant 1$.

**Corollary 2.** $S_i^1\subseteq S_{i+1}^1$, $i\geqslant 1$.

**Corollary 3.** $S_i^{1t}\subseteq S_{i+1}^{1t}$, $i\geqslant 1$.

**Lemma 19.** $p^{x^2}\in S_3^t-S_2^t$, $p>1$.

**Corollary.** (a) *For any n-ary function $f$ in $S_2^t$, there exists $c>1$ such that $f(x_1,\dots,x_n)\leqslant c^{\max(x_1,\dots,x_n,2)}$.*

(b) *For any n-ary function $f$ in $S_2^{1t}$, there exists $c\geqslant 1$ such that $f(x_1,\dots,x_n)\leqslant c\max(x_1,\dots,x_n,2)$.*

**Lemma 20.** $p^{x_1x_2}\in S_4^t-S_3^t$, $p>1$.

**Corollary.** (a) *For any n-ary function f in* $S_3^t$, *there exists* $c>1$ *and* $k\geqslant 1$ *such that* $f(x_1,\ldots,x_n)\leqslant c^{[\max(x_1,\ldots,x_n,2)]^k}$.

(b) *For any n-ary function f in* $S_3^{1t}$, *there exists* $k\geqslant 1$ *such that* $f(x_1,\ldots,x_n)\leqslant [\max(x_1,\ldots,x_n,2)]^k$.

**Lemma 21.** (a) *If an n-ary function* $f\in S_i$, $i\geqslant 4$, *then there exists* $k\geqslant 1$ *such that* $f(x_1,\ldots,x_n)\leqslant f_i(\max(x_1,\ldots,x_n,2),k)$ (*when* $f(x_1,\ldots,x_n)$ *is defined*).

(b) *If an n-ary function* $f\in S_i^1$, $i\geqslant 1$, *then there exists* $k\geqslant 1$ *such that* $f(x_1,\ldots,x_n)\leqslant f_i(\max(x_1,\ldots,x_n,2),k)$ (*when* $f(x_1,\ldots,x_n)$ *is defined*).

**Proof.** Similar to Lemma 10.

Note that Lemma 21(a) is not true for $i<4$. In fact, $p^x\in S_i$, but not bounded by $f_i(\max(x,2),k)$ for any $k\geqslant 1$, $i=1,2,3$, $(p>1.)$

**Lemma 22.** $f_i\in S_{i+1}^t-S_i^t$, $i\geqslant 4$.

**Proof.** Similar to Lemma 11, using Lemma 21.

From Theorem 5 and Corollary 1, and Lemmas 19, 20 and 22, we have

**Theorem 6.** $S_i\subset S_{i+1}$, $i\geqslant 2$.

**Corollary.** $S_i^t\subset S_{i+1}^t$, $i\geqslant 2$.

**Lemma 23.** $f_i\in S_{i+1}^{1t}-S_i^{1t}$, $i\geqslant 1$.

**Proof.** Similar to Lemma 11.

From Corollaries 2 and 3 of Theorem 5 and Lemma 23, we have

**Theorem 7.** $S_i^1\subset S_{i+1}^1$, $i\geqslant 1$.

**Corollary.** $S_i^{1t}\subset S_{i+1}^{1t}$, $i\geqslant 1$.

**Theorem 8.** $S_1=S_2$.

**Proof.** From Theorem 5, we have that $S_1\subseteq S_2$. To prove $S_2\subseteq S_1$, let an $n$-ary function $f\in S_2$. From Lemma 18, there exists an automaton $M$ and $k\geqslant 1$ such that $f$ is strongly computable in space $f_{2,k}^{(n)}$ by $M$. If $k=1$, clearly, we have $f\in S_1$. Assume that $k\geqslant 2$. Without loss of generality, we assume that $l_j=r_j=1$, $j=1,\ldots,m$. Denote $|\mathscr{A}_j|=p_j$, $j=1,\ldots,m$. We construct an automaton $M^*$ with $|\mathscr{A}_j^*|=|\mathscr{A}_{m+j}^*|=\sum_{c=1}^k p_j^c$ (denoting as $h_j$), $j=1,\ldots,m$, and $l_j^*=r_j^*=1$, $j=1,\ldots,m^*$. $M^*$ simulates $M$ with the $j$th tape of $M^*$ as the $j$th tape of $M$. $M$ behaves as follows:

(1) From the scan state of $M^*$ determine the scan state of $M$, say $\gamma_0$, and make $\rho(\gamma_0)$, $\eta_j(\gamma_0)=e_j$, $\lambda_j(\gamma_0)=x'_j$, $j=1,\ldots,m$.

(2) Let $x_j=\varphi_{p_j}^{-1}(\varphi_{h_j}((j)))$, $j=1,\ldots,m$.

$$\varphi_{h_j}^{-1}(\varphi_{p_j}((R_{-e_j}x_j)x'_j))\Rightarrow m+j, \quad j=1,\ldots,m.$$

(3) $m+j\Rightarrow j$, $j=1,\ldots,m$.

(4) If $M$ reaches a halting state, then halt; otherwise go to (1).

Since the tape-lengths of $M$ are at most $f_2(\max(x_1,\ldots,x_n,2),\ k)$ when $M$ computes $f(x_1,\ldots,x_n)$, the tape-lengths of $M^*$ are at most $\max(x_1,\ldots,x_n,2)$ when $M^*$ computes $f(x_1,\ldots,x_n)$. Therefore, $f_1\in S_1$.

**Corollary.** $S_1^t=S_2^t$.

## 4. Relationship among complexity classes. Hierarchy of PR($\theta$).

**Theorem 9.** $T_i=S_i$, $i\geq 4$; $T_i\subseteq S_i$, $i\geq 2$.

**Proof.** From Corollary of Lemma 15, we have that $T_i\subseteq S_i$, $i\geq 2$. Let $f$ be an $n$-ary function in $S_i$, $i\geq 4$. From Lemma 18, there exists an automaton $M$ and $k\geq 1$ such that $f$ is strongly computable in space $f_{i,k}^{(n)}$ by $M$. Denote $|\mathscr{A}_j|=p_j$, $j=1,\ldots,m$, and $w=\max(x_1,\ldots,x_n,2)$. Since the $j$th tape-lengths of $M$ are at most $f_i(w,k)$ during computing $f(x_1,\ldots,x_n)$, in case of $p_j>1$ the number of the $j$th external states is at most

$$\sum_{s=0}^{f_i(w,k)} p_j^s\leq(2p_j)^{f_i(w,k)}\leq f_3(f_2(w,p_j),f_i(w,k))\leq f_i(w,k+p_j),$$

and in case of $p_j=1$ the number of the $j$th external state is at most

$$f_i(w,k)+1\leq f_i(w,k)+f_i(w,1)\leq f_i(w,k+p_j).$$

Thus, the number of states

$$\leq t\prod_{j=1}^{m} f_j(w,k+p_j)\leq f_i(w,c)$$

$t$ being the number of internal states of $M$, and $c$ denoting $mk+p_1+\cdots+p_m+t$. So, within $f_i(w,c)$ steps a repeat state must occur. Therefore, the states enter in a circulation. Clearly, $f(x_1,\ldots,x_n)$ is defined if and only if the circulation consists of a halting state. Thus, $f$ is computable in time $f_{i,c}^{(n)}$ by $M$. Therefore, $f\in T_i$. We conclude that $S_i\subseteq T_i$, $i\geq 4$.

**Corollary 1.** $T_i^t=S_i^t$, $i\geq 4$; $T_i^t\subseteq S_i^t$, $i\geq 2$.

**Corollary 2.** $T_i^1=S_i^1$, $i\geq 3$; $T_2^1\subseteq S_2^1$.

**Corollary 3.** $T_i^{1t} = S_i^{1t}, i \geq 3$; $T_2^{1t} \subseteq S_2^{1t}$.

Applying usual method in automata theory, it is not hard to show the following results.

**Lemma 24.** (a) *$T_i$ is closed under the operations of substitution of a constant and identification of variables, $i \geq 2$.*

(b) *$T_i$ is closed under the operation of substitution of functions, $i \geq 4$; but $T_i$ is not, $i = 2, 3$.*

**Corollary 1.** *Replacing $T_i$ by $T_i^t$ in Lemma 24 the results hold.*

**Corollary 2.** *$T_i^1$ is closed under the operations of substitution of a constant, identification of variables and substitution of functions, $i \geq 2$.*

**Corollary 3.** *Replacing $T_i^1$ by $T_i^{1t}$ in Corollary 2 the results hold.*

**Lemma 25.** *$T_i^1$ is closed under the operation of limited resursion, $i \geq 3$.*

**Corollary.** *$T_i^{1t}$ is closed under the operation of limited recursion, $i \geq 3$.*

Let $U_1(x, y) = x$, $U_2(x, y) = y$, and

$$\theta(x) = \begin{cases} 0 & \text{if } x > 0, \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

From [7], Grzegorczyk's class $\mathscr{E}^i$ is the least class that includes $x+1$, $U_1(x, y)$, $U_2(x, y)$ and $f_i(x, y)$ and that is closed under the operations of substitution of a constant, identification of variables, substitution of functions and limited recursion. We use $\mathscr{E}^i(\theta)$ to denote the least class that includes $x+1$, $U_1(x, y)$, $U_2(x, y)$, $f_i(x, y)$ and $\theta(x)$ and that is closed under the operations of substitution of a constant, identification of variables, substitution of functions and limited recursion. ($i \geq 0$.)

Note that $f(x_1, \ldots, x_n)$ is defined only if all the $x_1, \ldots, x_n$ are defined.

**Lemma 26.** *$x+1$, $U_1(x, y)$, $U_2(x, y)$, and $\theta(x)$ are all in $T_i^1$, $i \geq 2$.*

From Corollary 1 of Lemma 4, Corollary 2 of Lemma 24 and Lemmas 25 and 26, we have

**Lemma 27.** $\mathscr{E}^i(\theta) \subseteq T_{i+1}^1$, $i \geq 2$.

From Corollary 2 of Lemma 4, Corollary 3 of Lemma 24, Corollary of Lemma 25 and Lemma 26, we have

**Lemma 28.** [9]. $\mathscr{E}^i \subseteq T^{1t}_{i+1}, i \geq 2$.

Represent the non-negative integers by their $p$-ary codes, and let $L_p(l, x)$, $R_p(-l, x)$, $\text{Con}_p(x, y)$ and $|x|_p$ be $L_lx$, $R_{-l}x$, $xy$ and $|x|$ respectively.

**Lemma 29.** $L_p(l, x), R_p(-l, x), \text{Con}_p(x, l) \in \mathscr{E}^1$, *$l$ being a non-negative integer.*

**Proof.** In case of $p=1$, $L_p(1, x) = 1 \dot{-} (1 \dot{-} x)$, $R_p(-1, x) = x \dot{-} 1$.[3] In case of $p>1$,

$$L_p(1, x) = \begin{cases} r(x, p) & \text{if } r(x, p) \neq 0, \\ p & \text{if } r(x, p) = 0 \text{ and } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$R_p(-1, x) = \begin{cases} q(x, p) & \text{if } r(x, p) \neq 0, \\ q(x, p) \dot{-} 1 & \text{if } r(x, p) = 0. \end{cases}$$

Obviously, $R_p(-i-1, x) = R_p(-1, R_p(-i, x))$, $i = 1, 2, \ldots$; $L_p(l, x) = c_1 + c_2p + \cdots + c_lp^{l-1}$, when $L_p(1, x) = c_1, L_p(1, R_p(-1, x)) = c_2, \ldots, L_p(1, R_p(-l+1, x)) = c_l$, $c_1, \ldots, c_i \in \{1, \ldots, p\}$, and $c_{i+1} = \cdots = c_l = 0$ for some $0 \leq i \leq l$.

In case of $p = 1$, $Con_p(x, c) = x + c$. In case of $p > 1$, Let $\psi_p(0) = 0$,

$$\psi_p(x+1) = \begin{cases} \psi_p(x) & \text{if } x < p\psi_p(x), \\ p\psi_p(x) + 1 & \text{if } x = p\psi_p(x). \end{cases}$$

Clearly, $\psi_p(x) \leq x$. Since $\psi_p(x) = 1 + p + \cdots + p^{|x|_p - 1} = (p^{|x|_p} - 1)/(p-1)$, we have that $p^{|x|_p} = (p-1)\psi_p(x) + 1$. Also, $\text{Con}_p(x, c) = x + cp^{|x|_p}$.

It is easy to see that all the functions mentioned above belong to $\mathscr{E}^1$.

Let $M$ be an automaton. Denote $|\mathscr{A}_j| = p_j$, $j = 0, \ldots, m$. We identify a word $x$ in $W(\mathscr{A}_i)$ with the integer $\varphi_{p_i}(x)$, then $\rho$, $\eta_j$, $\lambda_j$, $j = 1, \ldots, m$, and $\xi_M$ are functions over $N$ also. We extend $\xi_M$ as a total function such that $\xi_M(x_0, \ldots, x_m) = (x_0, \ldots, x_m)$ when $x_0 < \sum_{i=1}^q p_0^{i-1}$ or $x_0 > \sum_{i=1}^q p_0^i$, $q$ being the capacity of $M$.

**Lemma 30.** $\xi_M \in \mathscr{E}^1$, *i.e.* $\xi_i \in \mathscr{E}^1$, $i = 0, \ldots, m$, *$\xi_i$ being the $(i+1)$th component function of $\xi_M$.*

**Proof.** For any $x_0, \ldots, x_m$ in $N$, denote $\xi_M(x_0, \ldots, x_M) = (x'_0, \ldots, x'_m)$, $y_0 = x_0$, and $y_j = L_{p_j}(l_j, x_j)$, $j = 1, \ldots, m$. We have that

$$x'_0 = \rho(y_0, \ldots, y_m)$$

[3] $x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$

$$x'_j = \begin{cases} \mathrm{Con}_{p_j}(R_{p_j}(-l_j, x_j), \lambda_j(y_0, \ldots, y_m)) \\ \qquad\qquad \text{if } \eta_j(y_0, \ldots, y_m) = l_j \text{ and } \sum_{i=1}^{q} p_0^{i-1} \leqslant y_0 \leqslant \sum_{i=1}^{q} p_0^i, \\ \mathrm{Con}_{p_j}(x_j, \lambda_j(y_0, \ldots, y_m)) \\ \qquad\qquad \text{if } \eta_j(y_0, \ldots, y_m) = 0 \text{ and } \sum_{i=1}^{q} p_0^{i-1} \leqslant y_0 \leqslant \sum_{i=1}^{q} p_0^i, \\ x_j \qquad \text{otherwise,} \end{cases}$$

$j = 1, \ldots, m$.

From Lemma 29, it follows that $\xi_M \in \mathscr{E}^1$.

**Lemma 31.** $\xi_M^y(x_0, \ldots, x_m) \in \mathscr{E}^3$.

**Proof.** Denote $\xi_M^y(x_0, \ldots, x_m) = (\xi_0^y(x_0, \ldots, x_m), \ldots \xi_m^y(x_0, \ldots, x_m))$. Since $\mathscr{E}^2$ includes the pairing functions $(x+y)^2 + x$, $z \dot{-} \lfloor\sqrt{z}\rfloor^2$ and $\lfloor\sqrt{z}\rfloor \dot{-} (z \dot{-} \lfloor\sqrt{z}\rfloor^2)$,[4] we can construct an $(m+1)$-ary function $\tau$ and $m+1$ unary functions $\sigma_j$, $j = 0, 1, \ldots, m$, in $\mathscr{E}^2$ such that $\sigma_j(\tau(y_0, \ldots, y_m)) = y_j$, $j = 0, \ldots, m$, and $\tau$ is monotonously increasing (i.e. $\tau(x_0, \ldots, x_m) \leqslant \tau(y_0, \ldots, y_m)$ if $x_0 \leqslant y_0, \ldots, x_m \leqslant y_m$). Let

$$f(x_0, \ldots, x_m, y) = \tau(\xi_0^y(x_0, \ldots, x_m), \ldots, \xi_m^y(x_0, \ldots, x_m)).$$

Then we have $\xi_j^y(x_0, \ldots, x_m) = \sigma_j(f(x_0, \ldots, x_m, y))$, $j = 0, \ldots, m$. Therefore,

$$\begin{aligned} f(x_0, \ldots, x_m, 0) &= \tau(x_0, \ldots, x_m), \\ f(x_0, \ldots, x_m, y+1) &= \tau(\xi_0^{y+1}(x_0, \ldots, x_m), \ldots, \xi_m^{y+1}(x_0, \ldots, x_m)) = \\ &= \tau(\xi_0^1(\xi_0^y(x_0, \ldots, x_m), \ldots, \xi_m^y(x_0, \ldots, x_m)), \\ &\qquad \ldots, \xi_m^1(\xi_0^y(x_0, \ldots, x_m), \ldots, \xi_m^y(x_0, \ldots, x_m))) \\ &= \tau(\xi_0^1(\sigma_0(f(x_0, \ldots, x_m, y)), \ldots, \sigma_m(f(x_0, \ldots, x_m, y))), \\ &\qquad \ldots, \xi_m^1(\sigma_0(f(x_0, \ldots, x_m, y)), \ldots, \sigma_m(f(x_0, \ldots, x_m, y)))). \end{aligned}$$

From Lemma 30, it follows that $f$ is definable from functions in $\mathscr{E}^2$ by primitive recursion. Let

$$b_j(x_0, \ldots, x_m, y) = \begin{cases} x_j + p_j^{|x_j|_{p_j}} p_{p_j}^{r_j y+1} & \text{if } p_j > 1, \\ x_j + r_j y & \text{if } p_j = 1. \end{cases}$$

Clearly, $\xi_j^y(x_0, \ldots, x_m) \leqslant b_j(x_0, \ldots, x_m, y)$, $j = 1, \ldots, m$, and $\xi_0^y(x_0, \ldots, x_m) \leqslant x_0 + \sum_{i=1}^{q} p_0^i$. Since $\tau$ is monotonously increasing, we have that

$$f(x_0, \ldots, x_m, y) \leqslant \tau(x_0 + \sum_{i=1}^{q} p_0^i, b_1(x_0, \ldots, x_m, y), \ldots, b_m(x_0, \ldots, x_m, y)).$$

[4] $\lfloor u \rfloor$ denotes the greatest integer $\leqslant u$.

It is obvious that $b_j \in \mathscr{E}^3$, $j=1,\ldots,m$. Thus, $f \in \mathscr{E}^3$. It follows that $\xi_j^y(x_0,\ldots,x_m) \in \mathscr{E}^3$, $j=0,\ldots,m$.

Note that $b_j \in \mathscr{E}^1$ if $p_j=1$. We have

**Corollary.** *Let $M$ be a unary automaton. Then $\xi_M^y(x_0,\ldots,x_m) \in \mathscr{E}^2$.*

**Lemma 32.** $T_{i+1} \subseteq \mathscr{E}^i(\theta)$, $T_{i+1}^t \in \mathscr{E}^i$, $i \geq 3$.

**Proof.** Let $f$ be an $n$-ary function and $i \geq 3$. Assume that $f \in T_{i+1}$. Then there exists an automaton $M$ and $k \geq 1$ such that $f$ is computable in time $f_{i+1,k}^{(n)}$ by $M$. Let

$$g(x_0,\ldots,x_m,y)=\begin{cases}\xi_M^y(x_0,\ldots,x_m) & \text{if } \xi_M^y(x_0,\ldots,x_m) \text{ is a halting state,}\\ \text{undefined} & \text{otherwise.}\end{cases}$$

Let $h(x_0,\ldots,x_m)$ be the characteristic function of the following predicate:

$$\sum_{i=1}^{q} p_0^{i-1} \leq x_0 \leq \sum_{i=1}^{q} p_0^i \;\&\; \rho(\gamma)=x_0 \;\&\; \mathop{\&}_{j=1}^{m} [(x_j=0 \vee \eta_j(\gamma)=0) \;\&\; \lambda_j(\gamma)=\Lambda],$$

where $\gamma=(x_0, L_{p_1}(l_1,x_1),\ldots,L_{p_m}(l_m,x_m))$. Then, we have that

$$g(x_0,\ldots,x_m,y)=(\xi_0^y(x_0,\ldots,x_m)+\theta(1 \dot{-} h(\xi_M^y(x_0,\ldots,x_m))),$$
$$\ldots,\xi_m^y(x_0,\ldots,x_m)+\theta(1 \dot{-} h(\xi_M^y(x_0,\ldots,x_m,y)))).$$

It is easy to see that $h \in \mathscr{E}^1$. From Lemma 31, we have that $g \in \mathscr{E}^3(\theta)$. Denote the $(n+2)$th component function of $g$ by $g_{n+1}$. Then there exists $x_0$ in $N$ such that

$$f(x_1,\ldots,x_n)=g_{n+1}(x_0,\ldots,x_m,f_{i+1}(\max(x_1,\ldots,x_n,2),k)),$$

where $x_{n+1}=\cdots=x_m=0$. It follows that $f \in \mathscr{E}^i(\theta)$. We conclude that $T_{i+1} \in \mathscr{E}^i(\theta)$.

In the case that $f$ is a total function, replace 'undefined' by '$(0,\ldots,0)$' in the definition of $g$. Then we have that

$$g(x_0,\ldots,x_m,y)=(\xi_0^y(x_0,\ldots,x_m)(1 \dot{-} h(\xi_M^y(x_0,\ldots,x_m))),$$
$$\ldots,\xi_m^y(x_0,\ldots,x_m)(1 \dot{-} h(\xi_M^y(x_0,\ldots,x_m)))).$$

Thus, $g \in \mathscr{E}^3$. It follows $f \in \mathscr{E}^3$. Therefore $T_{i+1}^t \subseteq \mathscr{E}^i$.

From Corollary of Lemma 31 and the proof of Lemma 32, we have

**Corollary.** $T_{i+1}^1 \subseteq \mathscr{E}^i(\theta)$, $T_{i+1}^{1t} \subseteq \mathscr{E}^i$, $i \geq 2$.

**Theorem 10.**

$$T_{i+1}=\mathscr{E}^i(\theta),\; T_{i+1}^t=\mathscr{E}^i,\; i \geq 3;$$
$$T_{i+1}^1=\mathscr{E}^i(\theta),\; T_{i+1}^{1t}=\mathscr{E}^i,\; i \geq 2.$$ [5]

[5] From [9], it can be obtained that $T_{i+1}^{1t}=\mathscr{E}^i$, $i \geq 3$, $\mathscr{E}^2 \subseteq T_3^{1t}$.

**Proof.** When $i \geq 3$, from Lemmas 27 and 32, we have that $\mathscr{E}^i(\theta) \subseteq T^1_{i+1} \subseteq \mathscr{E}^i(\theta)$. It immediately follows that $T_{i+1} = \mathscr{E}^i(\theta)$. From Lemmas 28 and 32, we have that $\mathscr{E}^i \subseteq T^{1t}_{i+1} \subseteq T^t_{i+1} \subseteq \mathscr{E}^i$. It immediately follows that $T^t_{i+1} = \mathscr{E}^i$.

When $i \geq 2$, from Lemmas 27 and 28 and Corollary of Lemma 32, we have that $\mathscr{E}^i(\theta) \subseteq T^1_{i+1} \subseteq \mathscr{E}^i(\theta)$, $\mathscr{E}^i \subseteq T^{1t}_{i+1} \subseteq \mathscr{E}^i$. It immediately follows that $T^1_{i+1} = \mathscr{E}^i(\theta)$ and $T^{1t}_{i+1} = \mathscr{E}^i$.

Using Theorem 9 and Corollary, we have

**Corolloary.** (a) $T_{i+1} = T^1_{i+1} = S_{i+1} = S^1_{i+1} = \mathscr{E}^i(\theta)$, $i \geq 3$; $T^1_3 = S^1_3 = \mathscr{E}^2(\theta)$.
(b) $T^t_{i+1} = T^{1t}_{i+1} = S^t_{i+1} = S^{1t}_{i+1} = \mathscr{E}^i$, $i \geq 3$; $T^{1t}_3 = S^{1t}_3 = \mathscr{E}^2$.

It is easy to show that $S^1_3 \subset S_1$. But we show a stronger result. Let

$$S_0 = \bigcup_{\substack{n \geq 0 \\ p \geq 2}} \mathrm{SC}(\pi^{(n)}_p), \qquad S^t_0 = S_0 \bigcap \mathrm{TOL},$$

where $\pi^{(n)}_p(x_1, \ldots, x_n) = \max(\log_p x_1, \ldots, \log_p x_n, 1)$.

**Lemma 33.** *Let $p \geq 2$. For any $p_1, p_2 \geq 1$, there exists an automaton $M$ such that $|\mathscr{A}_j| = p_j$, $l_j = r_j = 1$, $j = 1, 2$, and $\lfloor \log_p x \rfloor$ is strongly computable in space $\max(|\lfloor \log_p x \rfloor|_{p_2}, |x|_{p_1})$ by $M$.*

**Proof.** We construct $M$ which behaves as follows:
(1) If $\langle 1 \rangle = \Lambda$, then enter in a loop, otherwise go to (2).
(2) $1 \Rightarrow 1$ & $3$.
(3) $a_1 \Rightarrow 4$.
(4) If $\varphi_{p_1}(\langle 3 \rangle) < p\varphi_{p_1}(\langle 4 \rangle)$, then halt, otherwise go to (5).
(5) $\varphi^{-1}_{p_2}(\varphi_{p_2}(\langle 2 \rangle) + 1) \Rightarrow 2$, $\varphi^{-1}_{p_1}(p\varphi_{p_1}(\langle 4 \rangle)) \Rightarrow 4$, go to (4).

**Corollary.** *Let $p \geq 2$. Then there exists a unary automaton $M$ such that $l_j = r_j = 1$, $j = 1, \ldots, m$, and $\lfloor \log_p x \rfloor$ is strongly computable in space $x$ by $M$.*

**Lemma 34.** *Let $f$ be an $n$-ary function in $S_0$. Then there exists an automaton $M$ and $p \geq 2$ such that $f$ is computable in space $\pi^{(n)}_p$ and when $M$ computes $f(x_1, \ldots, x_n)$ the tape-lengths are at most $w$ if $|x_j|_{p_j} \leq w$, $j = 1, \ldots, n$, where $w = \pi^{(n)}_p(x_1, \ldots, x_n)$, and $p_j = |\mathscr{A}_j|$, $j = 1, \ldots, n$.*

**Proof.** Similar to Lemma 18.

**Lemma 35.** (a) *$T^1_2$ contains functions $r(x, p)$, $q(x, p)$, $x(1 \dot{-} y)$, $x + y$, $x \dot{-} y$, $L_p(l, x)$, $R(-l, x)$ and* $\mathrm{Con}_p(x, l)$ $(p \geq 1, l \geq 0)$.

(b) *$T_2^1$ contains the characteristic functions of predicates $x < y$, $x \leqslant y$ and $x = y$, and the predicates of $T_2^1$ (i.e. the predicates of which the characteristic functions are in $T_2^1$) is closed under Boolean operations*

(c) *$T_2^1$ is closed under definition by cases.*

**Proof.** Obviously, the first 5 functions belong to $T_2^1$. Since characteristic functions of predicates $\neg(h(x_1, \ldots, x_n) = 0)$ and $(h(x_1, \ldots, x_n) = 0)$ & $(g(x_1, \ldots, x_n) = 0)$ are $1 \dot{-} h(x_1, \ldots, x_n)$ and $1 \dot{-} (1 \dot{-} (h(x_1, \ldots, x_n) + g(x_1, \ldots, x_n)))$ respectively, the predicates of $T_2^1$ are closed under Boolean operations. Part (b) is obtained from the fact that $1 \dot{-} (1 \dot{-} (x \dot{-} y))$ is the characteristic function of $x \leqslant y$, $x = y$ is equivalent to $x \leqslant y$ & $y \leqslant x$ and $x < y$ is equivalent to $\neg(y \leqslant x)$. Part (c) is obtained from the fact that $T_2^1$ contains $x(1 \dot{-} y)$ and $x + y$ and is closed under substitution of functions. Using (c), from the proof of Lemma 29, we have that $L_p(l, x)$, $R_p(-l, x) \in T_2^1$. Finally, we show that $\mathrm{Con}_p(x, l) \in T_2^1$. Since $\mathrm{Con}_p(x, l) = x + l$ if $p = 1$, and $\mathrm{Con}_p(x, l) = x + lp^{|x|_p}$ if $p > 1$, it is sufficient to show that $p^{|x|_p} \in T_2^1$ for $p > 1$. We construct a unary automaton $M$ with $l_j = r_j = 1$, $j = 1, \ldots, m$. $M$ behaves as follows:

(1) If $\langle 1 \rangle = \Lambda$, then $a_1 \Rightarrow 2$ and halt, otherwise go to (2).

(2) $a_1 \Rightarrow 3$.

(3) $3 \Rightarrow 4$, $\varphi_1^{-1}(\varphi_1(1) \dot{-} p\varphi_1(\langle 3 \rangle)) \Rightarrow 1$.

(4) If $\langle 1 \rangle = \Lambda$, then go to (6), otherwise go to (5).

(5) $\varphi_1^{-1}(p\varphi_1(\langle 4 \rangle)) \Rightarrow 3$, $\Lambda \Rightarrow 4$, and go to (3).

(6) $\varphi_1^{-1}(p\varphi_1(\langle 4 \rangle)) \Rightarrow 2$, and halt.

It is easy to show that $M$ computes $p^{|x|_p}$ and the number of steps is at most

$$1 + 1 + \sum_{i=1}^{|x|_p} [(p^i + 1) + 1 + (p^i + 1)] \leqslant 2 + 2px + 3|x|_p$$

$$\leqslant (2p + 4)\max(x, 2).$$

Therefore, $p^{|x|_p} \in T_2^1$. (Note that $p^{|x|_p}$ is computable in space $x + 1$ by $M$.)

**Corollary.** $|x|_p \in T_2^1 \cap S_1^1$.

**Proof.** Modify $M$ constructed above as follows: Cancel "$a_1 \Rightarrow 2$" in (1), add "$\mathrm{W}a_1 \Rightarrow 2$" to (3), and cancel "$\varphi_1^{-1}(p\varphi_1(\langle 4 \rangle)) \Rightarrow 2$" in (6). It can be shown that $|x|_p$ is computable in time $f_{2,2p+4}^{(1)}$ (or in space $f_{1,0}^{(1)}$) by $M$.

Using Lemma 35, from the proof of Lemma 30, we have

**Lemma 36.** $\xi_M \in T_2^1$.

From Lemma 35, we have

**Lemma 37.** *The characteristic function of the predicate "$(x_0, \ldots, x_m)$ is a halting state of $M$" belongs to $T_2^1$.*

**Theorem 11.** $S_3^1 = S_0 \subset S_1$.

**Proof.** Assume that an $n$-ary function $f \in S_3^1$. From Corollary of Lemma 18, there exists a unary automaton $M$ and $k \geq 1$ such that $f$ is strongly computable in space $f_{3,k}^{(n)}$ by $M$. Without loss of generality, we assume that $r_1 = \cdots = r_m = 1$. We construct an automaton $M^*$ with $m^* = m$ and $|\mathscr{A}_j^*| = p = 4^k$, $j = 1, \ldots, m^*$. $M^*$ simulates $M$ as follows: If the length of the $j$th external state of $M$ increases (or decreases) $c$ in a step, then the content of $j$th tape of $M^*$ increases (or decreases) $c$ in the corresponding step. Denote $w = \max(x_1, \ldots, x_n, 2)$. Since

$$f_3(w, k) = w^k = p^{\log_p w^k} = p^{\log_4 w} = p^{\frac{1}{2}\log_2 w} \leq p^{\lfloor \log_2 w \rfloor},$$

the tape-lengths are at most $\log_2 w$ when $M^*$ computes $f(x_1, \ldots, x_n)$ Thus, $f \in S_0$. We conclude that $S_3^1 \subseteq S_0$. Also, from Lemma 7, $f$ is strongly computable in space $f_{1,0}^{(n)}$. Thus $f \in S_1$. We conclude that $S_3^1 \subseteq S_1$.

On the other hand, assume that an $n$-ary function $f \in S_0$. From Lemma 34, there exists an automaton $M$ and $p \geq 2$ such that $f$ is computable in space $\pi_p^{(n)}$, and when $M$ computes $f(x_1, \ldots, x_n)$ the tape-lengths are at most $u$ if $|x_j|_{p_j} \leq u$, where $u = \pi_p^{(n)}(x_1, \ldots, x_n)$, and $|\mathscr{A}_j| = p_j$, $j = 1, \ldots, m$. Denote the component functions of $\xi_M$ by $\xi_0, \ldots, \xi_m$. Denote the characteristic function of the predicate "$(y_1, \ldots, y_{m+1})$ is a halting state of $M$" by $\xi_{m+1}(y_1, \ldots, y_{m+1})$. From Lemmas 36 and 37, we have that $\xi_i \in T_2^1$, $i = 0, \ldots, y_{m+1})$. From Lemmas 36 and 37, we have that $\xi_i \in T_2^1$, $i = 0, \ldots, m+1$. It follows that $\xi_i \in S_2^1$, $i = 0, \ldots, m+1$. Thus, for each $i$, $0 \leq i \leq m+1$, there exists a unary automaton $M_i$ and $k_i \geq 1$ such that $l_{ji} = r_{ji} = 1$, $j = 1, \ldots, m+2$, and $\xi_i$ is strongly computable in space $f_{2,k_i}^{(m+1)}$ by $M_i$. We construct a unary automaton $M^*$ with $l_j^* = r_j^* = 1$, $j = 1, \ldots, n+1$. $M^*$ behaves as follows:

(1) $j \Rightarrow j \ \& \ (n+3+m+j)$, $j = 1, \ldots, n$, and $c \Rightarrow n+3+m$, $c$ being the unary code of the initial internal state of $M$.

(2) Compute $u' = \max(\lfloor \log_p x_1 \rfloor, \ldots, \lfloor \log_p x_n \rfloor, 1)$ and $u'' = \max(|x_1|_{p_1}, \ldots, |x_n|_{p_n})$, $x_j$ being $\varphi_1(\langle j \rangle)$, $j = 1, \ldots, n$. If $u' < u''$, then enter in a loop, otherwise go to (3).

(3) $n+3+m+j \Rightarrow n+2+j$, $j = 0, \ldots, m$.

(4) Simulate $M_{m+1}$ with the $(n+2)$th, $\ldots$, $(n+2+m)$th, $(n+4+2m)$th tapes of $M^*$ as the 1st, $\ldots$, $(m+2)$th tapes of $M_{m+1}$ respectively, go to (5) as soon as $M_{m+1}$ reaches a halting state.

(5) If $\langle n+4+2m \rangle = \Lambda$, then go to (6), otherwise go to (7).

(6) $2n+3 \Rightarrow n+1$, and halt.

(7) Simulate $M_i$ with the $(n+2)$th, $\ldots$, $(n+2+m)$th, $(n+3+m+i)$th tapes of $M^*$ as the 1st, $\ldots$, $(m+2)$th tapes of $M_i$ respectively, $i = 0, \ldots, m$; go to (3).

It is evident that $u'' \leq u'$ if an only if $u'' \leq u$. Thus, when $u'' > u$ no halting state of $M^*$ can be reached. Let $u'' < u$. Clearly, $M$ computes $f(x_1, \ldots, x_n)$, and for any $j$, $1 \leq j \leq m$, we have that $|\varphi_1(\langle n+2+j \rangle)|_{p_j} \leq u$. It follows that

$$\begin{aligned} \varphi_1(\langle n+2+j \rangle) &\leq 2p_j^u \leq 2p_j^{\log_p \max(x_1, \ldots, x_n)+1} \\ &\leq 2p_j[\max(x_1, \ldots, x_n)]^{\log_p p_j}. \end{aligned}$$

Since the $(n+2)$th external state of $M^*$ simulates the internal state of $M$, there exists a constant $e$ such that $\varphi_1(\langle n+2\rangle)\leqslant e$. Using Lemma 33 and Corollary of Lemma 35, we can show that the tape-lengths are at most

$$\max[k_ie,\ 2k_ip_j(\max(x_1,\dots,x_n))^{\log_p p_i},\ i=0,\dots,m,$$
$$j=1,\dots,m+1,\ \max(x_1,\dots,x_n,c)]$$

Thus there exists $k\geqslant 1$ such that the tape-lengths are at most $[\max(x,\dots,x,2)]^k$. It follows that $f\in S_3^1$. We conclude that $S_0\subseteq S_3^1$. Therefore $S_0=S_3^1$.

Suppose to the contrary that $p^x\in S_3^1$, $p>1$. From Lemma 21, there exists $k\geqslant 1$ such that $p^x\leqslant f_3(\max(x,2),k)$. It follows that $p^x\leqslant x^k$ if $x\geqslant 2$, this contradicts to $\lim_{x\to\infty}p^x/x^k=\infty$. We conclude that $p^x\notin S_3^1$. It is obvious that $p^x\in S_1$. Thus $S_3^1\subset S_1$.

**Corollary.** $S_3^{1t}=S_0^t\subset S_1^t$.

The results obtained hitherto can be summarized in following diagram:

$$\begin{array}{cccccccccccccccc}
 &&&&&&&&&&&& S_4^1 && S_5^1 & \dots\\
 &&&&&&&&&&&& \parallel && \parallel & \\
S_1^1 & \subset & S_2^1 & \subset & S_3^1 & =S_0\subset & S_1 & = & S_2 & \subset & S_3 & \subset & S_4 & \subset & S_5 & \subset\dots\\
 &&&&&&&& \rotatebox{90}{$\subseteq$} && \rotatebox{90}{$\subseteq$} && \parallel && \parallel & \\
 && \rotatebox{90}{$\subseteq$} && \parallel && T_1 & = & T_2 & \subset & T_3 & \subset & T_4 & \subset & T_5 & \subset\dots\\
 &&&&&&&&&&&& \parallel && \parallel & \\
T_1^1 & = & T_2^1 & \subset & T_3^1 &&&&&&& \subset & T_4^1 & \subset & T_5^1 & \subset\dots\\
 &&&& \parallel &&&&&&&& \parallel && \parallel & \\
 &&&& \mathscr{E}^2(\theta) &&&&&&&& \mathscr{E}^3(\theta) && \mathscr{E}^4(\theta) & \dots
\end{array}$$

For total functions, an analogous diagram can be made.

Denote the class of functions which are primitive recursive in $\theta$ by $\mathrm{PR}(\theta)$. It is obvious that $\mathscr{E}^i(\theta)\subset\mathrm{PR}(\theta)$. Thus $\bigcup_{i=1}^\infty\mathscr{E}(\theta)\subseteq\mathrm{PR}(\theta)$. To show that $\mathrm{PR}(\theta)\subseteq\bigcup_{i=1}^\infty\mathscr{E}^i(\theta)$, we first generalize a lemma in [9].

We define recursively $\mathscr{R}_i(\theta)$, $i=0,1,\dots$, as follows:

(1) $x+1$, $U_1(x,y)$, $U_2(x,y)$, and $\theta(x)$ are in $\mathscr{R}_i(\theta)$.

(2) The functions which are defined from functions in $\mathscr{R}_i(\theta)$ by the operations of substitution of a constant, identification of variables and substitution of functions are in $\mathscr{R}_i(\theta)$.

(3) The functions which are defined by primitive recursion from functions in $\mathscr{R}_{i-1}(\theta)$ are in $\mathscr{R}_i(\theta)$ $(i\geqslant 1)$.

(4) Nothing is in $\mathscr{R}_i$ unless its being there follows from (1)–(3).

**Lemma 38.** $\mathscr{R}_i(\theta)\subseteq\mathscr{E}^{i+1}(\theta)$, $i\geqslant 0$.

**Proof.** By induction on $i$.

*Basis.* $i=0$, 1. It is easy to show that the functions in $\mathscr{R}_0(\theta)$ are of the form of $x+c$, $c$, $\theta(x)+c$ and $\emptyset(x_1,\dots,x_n)$, $c=0,1,\dots$, where the domain of $\emptyset$ is the empty set. It follows that $\mathscr{R}_0(\theta) \subseteq \mathscr{E}^0(\theta)$. Therefore $\mathscr{R}_0(\theta) \subseteq \mathscr{E}^1(\theta)$. By exhausting the possibilities, we can show that the functions which are defined by primitive recursion from functions in $\mathscr{R}_0(\theta)$ are of the following forms:

$x+z+c$,

$x+c+dz$,

$(x+c)(1 \dot- y)+d(1 \dot- (1 \dot- y))$,

$c(1 \dot- y)+(z+d)(1 \dot- (1 \dot- y))$,

$c+dy$,

$c(1 \dot- y)+d(1 \dot- (1 \dot- y))$,

$(x+c)(1 \dot- y)+(\theta(x)+d)(1 \dot- (1 \dot- y))$,

$(x+c+1)(1 \dot- y)+(d+1)(1 \dot- (1 \dot- y))$,

$\theta(2 \dot- y)+(x+c+1)(1 \dot- y)$,

$x(1 \dot- y)+(\theta(x)+d+1)(1 \dot- (1 \dot- y))$,

$\theta(2 \dot- y)+\theta(x)(1 \dot- (1 \dot- y))+x(1 \dot- y)$,

$(\theta(x)+c)(1 \dot- y)+(z+d)(1 \dot- (1 \dot- y))$,

$\theta(x)+c+dy$,

$c(1 \dot- y)+(\theta(z)+d)(1 \dot- (1 \dot- y))$,

$(c+1)(1 \dot- y)+(d+1)(1 \dot- (1 \dot- y))$,

$\theta(2 \dot- y)+(c+1)(1 \dot- y)$,

$\theta(1 \dot- y)+c$

$(\theta(x)+c)(1 \dot- y)+d(1 \dot- (1 \dot- y))$,

$(\theta(x)+c)(1 \dot- y)+(\theta(z)+d)(1 \dot- (1 \dot- y))$,

$\theta(x)+(c+1)(1 \dot- y)+(d+1)(1 \dot- (1 \dot- y))$,

$\theta(x)+\theta(2 \dot- y)+(c+1)(1 \dot- y)$,

$\theta(x)+\theta(1 \dot- y)$,

$x+c+\theta(1 \dot- y)$,

$c+\theta(1 \dot- y)$,

$\theta(x)+c+\theta(1 \dot- y)$.

$c, d = 0, 1, \ldots$, and $\emptyset$, where $z$ is $x$, $y$ or another variable. It is obvious that all the functions are in $\mathscr{E}^1(\theta)$. It follows that $\mathscr{R}_1(\theta) \subseteq \mathscr{E}^1(\theta)$. Therefore, $\mathscr{R}_1(\theta) \subseteq \mathscr{E}^2(\theta)$.

*Induction step.* Assume that $\mathscr{R}_i(\theta) \subseteq \mathscr{E}^{i+1}(\theta)$ for some $i \geqslant 1$. To prove that $\mathscr{R}_{i+1}(\theta) \subseteq \mathscr{E}^{i+2}(\theta)$, we first observe that $x+1$, $U_1(x, y)$, $U_2(x, y)$ and $\theta(x)$ are in $\mathscr{R}_{i+1}(\theta)$. Next, let $g$ and $h$ be in $\mathscr{R}_{i+1}(\theta)$ from which the function $f$ is obtained by substitution of a constant, or by identification of variables, or by substitution of functions. If $g$ and $h$ are in $\mathscr{E}^{i+2}(\theta)$, then $f$ is in $\mathscr{E}^{i+2}(\theta)$. Finally, let

$$f(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n),$$

$$f(x_1, \ldots, x_n, y+1) = h(x_1, \ldots, x_n, y, f(x_1, \ldots, x_n, y)), \; g, h \in \mathscr{R}_i(\theta).$$

From the induction hypothesis, we have that $g, h \in \mathscr{E}^{i+1}(\theta)$. From $T^1_{i+2} = \mathscr{E}^{i+1}(\theta)$, using Lemma 10, there exists $k \geqslant 2$ such that $h(x_1, \ldots, x_{n+2}) \leqslant f_{i+2}(\max(x_1, \ldots, x_{n+2}, 2), k)$. By induction on $y$, we can show that

$$f(x_1, \ldots, x_n, y) \leqslant f_{i+2}(\max(x_1, \ldots, x_n, g(x_1, \ldots, x_n), 2), k^y)$$

as shown in [9]. From $k^y \in \mathscr{E}^3$, it follows that

$$f_{i+2}(\max(x_1, \ldots, x_n, g(x_1, \ldots, x_n), 2), k^y) \in \mathscr{E}^{i+2}(\theta).$$

We conclude that $f \in \mathscr{E}^{i+2}(\theta)$.

From Lemma 38, we have that $\mathrm{PR}(\theta)=\bigcup_{i=0}^{\infty}\mathscr{R}_i(\theta)\subseteq\bigcup_{i=1}^{\infty}\mathscr{E}^i(\theta)$. This completes the proof of the following theorem.

**Theorem 12.** $\mathrm{PR}(\theta)=\bigcup_{i=1}^{\infty}\mathscr{E}^i(\theta)$.

Note that the results in [9] can be generalized to contain partial functions, that is

$$\mathscr{R}_i^{\mathrm{sim}}(\theta)=\mathscr{E}^{i+1}(\theta),\quad i\geqslant 2,\qquad \mathscr{R}_i(\theta)=\mathscr{E}^{i+1}(\theta),\quad i\geqslant 3,$$

$$\mathscr{R}_i^{\mathrm{sim}}(\theta)\subseteq\mathscr{E}^i(\theta),\quad i=0,1,\qquad \mathscr{R}_2^{\mathrm{sim}}(\theta)\nsubseteq\mathscr{E}^2(\theta),$$

where the definition of $\mathscr{R}_i^{\mathrm{sim}}(\theta)$ is similar to $\mathscr{R}_i(\theta)$ but instead of primitive recursion by simultaneously primitive recursion.

In the rest of this segment, we discuss the time and space closed properties of complexity classes.

**Lemma 39.** *If $g\in T_i$ and $i\geqslant 4$, then* $\mathrm{TC}(g)\subseteq T_i$.

**Proof.** Similar to Lemma 6, using Lemma 10.

**Corollary 1.** *If $g\in T_i^t$ and $i\geqslant 4$, then* $\mathrm{TC}^t(g)\subseteq T_i^t$.

**Corollary 2.** *If $g\in T_i^1$ and $i\geqslant 2$, then* $\mathrm{TC}^1(g)\subseteq T_i^1$.

**Corollary 3.** *If $g\in T_i^{1t}$ and $i\geqslant 2$, then* $\mathrm{TC}^{1t}(g)\subseteq T_i^{1t}$.

**Lemma 40.** *If $g\in S_i$ and $i\geqslant 4$, then* $\mathrm{SC}(g)\subseteq S_i$.

**Proof.** Similar to Lemma 18, using Lemma 21.

**Corollary 1.** *If $g\in S_i^t$* and *$i\geqslant 4$, then* $\mathrm{SC}^t(g)\subseteq S_i^t$.

**Corollary 2.** *If $g\in S_i^1$ and $i\geqslant 1$, then* $\mathrm{SC}^1(g)\subseteq S_i^1$.

**Corollary 3.** *If $g\in S_i^{1t}$ and $i\geqslant 1$, then* $\mathrm{SC}^{1t}(g)\subseteq S_i^{1t}$.

**Theorem 13.**

(a) $$\bigcup_{g\in T_i}\mathrm{TC}(g)=T_i,\qquad \bigcup_{g\in S_i}\mathrm{SC}(g)=S_i,$$

$$\bigcup_{g\in T_i^t}\mathrm{TC}^t(g)=T_i^t,\qquad \bigcup_{g\in S_i^t}\mathrm{SC}^t(g)=S_i^t,\qquad i\geqslant 4.$$

(b) $$\bigcup_{g\in T_i^1}\mathrm{TC}^1(g)=T_i^1,\qquad \bigcup_{g\in S_i^1}\mathrm{SC}^1(g)=S_i^1,$$

$$\bigcup_{g\in T_i^{1t}}\mathrm{TC}^{1t}(g)=T_i^{1t},\qquad \bigcup_{g\in S_i^{1t}}\mathrm{SC}^{1t}(g)=S_i^{1t},\qquad i\geqslant 1.$$

**Proof.** (a) Let $i \geq 4$. From Lemma 39, we have that $\bigcup_{g\in T_i} \mathrm{TC}(g) \subseteq T_i$. Since $T_i = \bigcup_{n\geq 0, k\geq 1} \mathrm{TC}(f_{i,k}^{(n)})$ and $f_{i,k}^{(n)} \in T_i$, we have that $T_i \subseteq \bigcup_{g\in T_i} \mathrm{TC}(g)$. Thus, $\bigcup_{g\in T_i} \mathrm{TC}(g) = T_i$. Therefore, $\bigcup_{g\in T_i^t} \mathrm{TC}^t(g) = T_i^t$.

The others in (a) can be shown analogously (using Lemma 40).

(b) Proof is simialar to (a), using Theorem 4.

**Corollary.**

(a)
$$\bigcup_{g\in\mathscr{E}^i(\theta)} \mathrm{TC}(g) = \bigcup_{g\in\mathscr{E}^i(\theta)} \mathrm{SC}(g) = \mathscr{E}^i(\theta).$$

$$\bigcup_{g\in\mathscr{E}^i} \mathrm{TC}^t(g) = \bigcup_{g\in\mathscr{E}^i} \mathrm{SC}^t(g) = \mathscr{E}^i, \quad i \geq 3.$$

(b)
$$\bigcup_{g\in\mathscr{E}^i(\theta)} \mathrm{TC}^1(g) = \bigcup_{g\in\mathscr{E}^i(\theta)} \mathrm{SC}^1(g) = \mathscr{E}^i(\theta),$$

$$\bigcup_{g\in\mathscr{E}^i} \mathrm{TC}^{1t}(g) = \bigcup_{g\in\mathscr{E}^i} \mathrm{SC}^{1t}(g) = \mathscr{E}^i, \quad i \geq 2.$$

(c)
$$\bigcup_{g\in\mathrm{PR}(\theta)} \mathrm{TC}(g) = \bigcup_{g\in\mathrm{PR}(\theta)} \mathrm{TC}^1(g) = \bigcup_{g\in\mathrm{PR}(\theta)} \mathrm{SC}(g) = \bigcup_{g\in\mathrm{PR}(\theta)} \mathrm{SC}^1(g) = \mathrm{PR}(\theta),$$

$$\bigcup_{g\in\mathrm{PR}} \mathrm{TC}^t(g) = \bigcup_{g\in\mathrm{PR}} \mathrm{TC}^{1t}(g) = \bigcup_{g\in\mathrm{PR}} \mathrm{SC}^t(g) = \bigcup_{g\in\mathrm{PR}} \mathrm{SC}^{1t}(g) = \mathrm{PR},$$

where PR denotes the class of primitive recursive functions.

## 5. Computation with time/space bounded by superexponential functions

For any $p \geq 2$ and $k \geq 1$, define $\exp_{p,k}$ as follows:

$$\exp_{p,1}(x) = p^x, \qquad \exp_{p,i+1}(x) = p^{\exp_{p,i}(x)}, \quad i \geq 1.$$

From the definition, we can show the following results.

**Lemma 41.**

(1) $y\exp_{p,k}(x) \leq \exp_{p,k}(x+y)$.

(2) *If* $r \leq \exp_{p,k}(1)$, *then* $rx \leq \exp_{p,k+1}(x)$.

(3) $f_4(x, k) \leq \exp_{p,k}(kx)$.

(4) *If* $k \leq \exp_{p,r-1}(1)$, *then* $f_4(x, k) \leq \exp_{p,k+r}(x)$.

(5) $\exp_{a,k}(x) + \exp_{b,k}(x) \leq \exp_{a+b,k}(x)$ *for* $x > 0$.

(6) $b\exp_{a,k}(x) \leq \exp_{ab,k}(x)$ *for* $x > 0$, $b > 0$.

(7) $(\exp_{a,k}(x))(\exp_{b,k}(x)) \leq \exp_{ab,k}(x)$.

For $p \geq 2$, $k \geq 1$ and $n \geq 0$, let

$$g_{p,k}^{(n)}(x_1, \ldots, x_n) = \exp_{p,k}(\max(x_1, \ldots, x_n, 1)).$$

Denote

$$T_{4,k} = \bigcup_{\substack{n\geq 0\\ p\geq 2}} \mathrm{TC}(g_{p,k}^{(n)}), \qquad T_{4,k}^1 = \bigcup_{\substack{n\geq 0\\ p\geq 2}} \mathrm{TC}^1(g_{p,k}^{(n)}),$$

$$S_{4,k} = \bigcup_{\substack{n\geq 0\\ p\geq 2}} \mathrm{SC}(g_{p,k}^{(n)}), \qquad S_{4,k}^1 = \bigcup_{\substack{n\geq 0\\ p\geq 2}} \mathrm{SC}^1(g_{p,k}^{(n)}),$$

$$T_{4,k}^t = T_{4,k} \cap \mathrm{TOL}, \qquad T_{4,k}^{1t} = T_{4,k}^1 \cap \mathrm{TOL},$$

$$S_{4,k}^t = S_{4,k} \cap \mathrm{TOL}, \qquad S_{4,k}^{1t} = S_{4,k}^1 \cap \mathrm{TOL}.$$

**Lemma 42.** *For any $p \geq 2$, there exists a unary automaton $M$ such that $l_j = r_j = 1$, $j = 1, \ldots, m$, and $p^x$ is strongly computable in time $8p^x$ by $M$.*

**Lemma 43.** *For any $k$, $p_1$, $p_2 \geq 1$ and $p \geq 2$, there exists an automaton $M$* and $r \geq 2$ *such that $|\mathcal{A}_j| = p_j$, $l_j = r_j = 1$, $j = 1, 2$, and* $\exp_{p,k}$ *is strongly computable in time $g_{r,k}^{(1)}$ by $M$.*

**Corollary 1.** *For any $k \geq 1$ and $p \geq 2$, there exists a unary automaton $M$ and $r \geq 2$ such that $l_j = r_j = 1$, $j = 1, 2$, and* $\exp_{p,k}$ *is strongly computable in time $g_{r,k}^{(1)}$ by $M$.*

**Corollary 2.** $\exp_{p,k+1} \in T_{4,k}$, $k \geq 1$, $p \geq 2$.

**Lemma 44.** *If an $n$-ary function $f \in T_{4,k}$, $k \geq 1$, then for any $p_1, \ldots, p_n \geq 1$, there exists an automaton $M$ and $p \geq 2$ such that $|\mathcal{A}_j| = p_j$, $l_j = r_j = 1$, $j = 1, \ldots, n$, and $f$ is strongly computable in time $g_{p,k}^{(n)}$ by $M$.*

**Proof.** Similar to Lemma 6.

**Corollary.** *If an $n$-ary function $f \in T_{4,k}^1$, $k \geq 1$, then there exists a unary automaton $M$ and $p \geq 2$ such that $l_j = r_j = 1$, $j = 1, \ldots, n$, and $f$ is strongly computable in time $g_{p,k}^{(n)}$ by $M$.*

**Lemma 45.** *If an $n$-ary function $f \in T_{4,k}$, $k \geq 1$, then there exists $p \geq 2$ such that*

$$f(x_1, \ldots, x_n) \leq \exp_{p,k+1}(\max(x_1, \ldots, x_n, 1))$$

*(when $f(x_1, \ldots, x_n)$ is defined).*

(b) *If an $n$-ary function $f \in T_{4,k}^1$, $k \geq 1$, then there exists $p \geq 2$ such that*

$$f(x_1, \ldots, x_n) \leq \exp_{p,k}(\max(x_1, \ldots, x_n, 1))$$

*(when $f(x_1, \ldots, x_n)$ is defined).*

**Proof.** Similar to Lemma 10.

**Theorem 14.** $T_{4,k} \subset T_{4,k+1}$, $T^1_{4,k} \subset T^1_{4,k+1}$, $k \geqslant 1$.

**Proof.** Let $f$ be an $n$-ary function in $T_{4,k}$. From Lemma 44, there exists an automaton $M$ and $p \geqslant 2$ such that $f$ is strongly computable in time $g^{(n)}_{p,k}$ by $M$. From $g^{(n)}_{p,k}(x_1, \ldots, x_n) \leqslant g^{(n)}_{p,k+1}(x_1, \ldots, x_n)$, using Lemma 7, $f$ is strongly computable in time $g^{(n)}_{p,k+1}$ by $M$. Therefore $f \in T_{4,k+1}$. We conclude that $T_{4,k} \subseteq T_{4,k+1}$.

Suppose to the contrary that $\exp_{2,k+2}(x) \in T_{4,k}$. From Lemma 45(a), there exists $p \geqslant 2$ such that $\exp_{2,k+2}(x) \leqslant \exp_{p,k+1}(\max(x, 1))$. This contradicts to $\lim_{x\to\infty} \exp_{2,k+2}(x)/\exp_{2,k+1}(x) = \infty$. Thus $\exp_{2,k+2}(x) \notin T_{4,k}$. But $\exp_{2,k+2}(x) \in T_{4,k+1}$ from Corollary 2 of Lemma 43, therefore it follows that $T_{4,k} \subset T_{4,k+1}$.

Using Corollary of Lemma 44, it can be analogously shown that $T^1_{4,k} \subseteq T^1_{4,k+1}$. Using Corollary 1 of Lemma 43 and Lemma 45(b), it can be analogously shown that $\exp_{2,k+1}(x) \in T^1_{4,k+1} - T^1_{4,k}$. It thus follows that $T^1_{4,k} \subset T^1_{4,k+1}$.

**Corollary.** $T^t_{4,k} \subset T^t_{4,k+1}$, $T^{1t}_{4,k} \subset T^{1t}_{4,k+1}$, $k \geqslant 1$.

**Theorem 15.** $T_4 = \bigcup_{k \geqslant 1} T_{4,k}$, $T^1_4 = \bigcup_{k \geqslant 1} T^1_{4,k}$.

**Proof.** Let an $n$-ary function $f \in T_4$. From Lemma 6, there exists an automaton $M$ and $k \geqslant 1$ such that $f$ is strongly computable in time $f^{(n)}_{4,k}$. Assume that $k \leqslant \exp_{2,r-1}(1)$. From Lemma 41, we have that

$$f_4(\max(x, 2), k) \leqslant \exp_{2,k+r}(\max(x, 2)) \leqslant \exp_{2,k+r+1}(\max(x, 1)).$$

From Lemma 7, $f$ is strongly computable in time $g^{(n)}_{2,k+r+1}$ by $M$. Thus $f \in T_{4,k+r+1} \subseteq \bigcup_{i \geqslant 1} T_{4,i}$. We conclude that $T_4 \subseteq \bigcup_{k \geqslant 1} T_{4,k}$.

Contrarily, let an $n$-ary function $f \in T_{4,k}$, $k \geqslant 1$. From Lemma 44, there exists an automaton $M$ and $p \geqslant 2$ such that $f$ is strongly computable in time $g^{(n)}_{p,k}$ by $M$. Take $c \geqslant 1$ such that $\exp_{p,k}(x) \leqslant f_4(x, c)$ if $2 \leqslant x < p$. From $\exp_{p,k}(x) \leqslant f_4(x, k+1)$ if $x \geqslant p$, it follows that $\exp_{p,k}(x) \leqslant f_4(x, \max(c, k+1))$ if $x \geqslant 2$. Thus $\exp_{p,k}(\max(x_1, \ldots, x_n, 1)) \leqslant f_4(\max(x_1, \ldots, x_n, 2), \max(c, k+1))$. From Lemma 7, $f$ is strongly computable in time $f^{(n)}_{4,\max(c,k+1)}$ by $M$. Therefore $f \in T_4$. It follows that $T_{4,k} \subseteq T_4$, for any $k \geqslant 1$. Then $\bigcup_{k \geqslant 1} T_{4,k} \subseteq T_4$. We conclude that $T_4 = \bigcup_{k \geqslant 1} T_{4,k}$.

Analogously, we can show that $T^1_4 = \bigcup_{k \geqslant 1} T^1_{4,k}$.

**Corollary.** $T^t_4 = \bigcup_{k \geqslant 1} T^t_{4,k}$, $T^{1t}_4 = \bigcup_{k \geqslant 1} T^{1t}_{4,k}$.

Using Lemma 6, Corollaries 1 and 2 of Lemma 43 and Corollary of Lemma 9, we can show the following results.

**Theorem 16.** $T_3 \subset T_{4,1}$, $T^1_3 \subset T^1_{4,1}$.

**Corollary.** $T^t_{4,1} \subset T^t_{1,4}$, $T^{1t}_3 \subset T^{1t}_{4,1}$.

**Theorem 17.** $T^1_{4,k} \subset T_{4,k} \subseteq T^1_{4,k+1}$, $k \geqslant 1$.

**Proof.** In the proof of Theorem 14, we have pointed out that $\exp_{2,k+1}(x) \notin T^1_{4,k}$. From Corollary 2 of Lemma 43, we have that $\exp_{2,k+1}(x) \in T_{4,k}$. It is obvious that $T^1_{4,k} \subseteq T_{4,k}$. Thus $T^1_{4,k} \subset T_{4,k}$.

Proving $T_{4,k} \subseteq T^1_{4,k+1}$ is similar to the proof of $S_0 \subseteq S^1_3$ in Theorem 11.

**Corollary.** $T^{1t}_{4,k} \subset T^t_{4,k} \subseteq T^t_{4,k+1}$, $k \geqslant 1$.

Similar to Lemma 24, we have

**Theorem 18.** (a) *Both $T_{4,k}$ and $T^1_{4,k}$ are closed under the operations of substitution of a constant, identification of variables and exchanges of variables, $k \geqslant 1$.*

(b) *Neither $T_{4,k}$ nor $T^1_{4,k}$ is closed under the operation of substitution of functions, $k \geqslant 1$.*

(c). *Let $h(x_1, \ldots, x_{n-1}, y_1, \ldots, y_t) = f(x_1, \ldots, x_{r-1}, g(y_1, \ldots, y_t), x_r, \ldots x_{n-1})$. If $g \in T_{4,k_1}$ and $f \in T_{4,k_2}$, then $h \in T_{4,k_1+k_2+1}$. If $g \in T^1_{4,k_1}$ and $f \in T^1_{4,k_2}$, then $h \in T^1_{4,k_1+k_2}$, $k_1, k_2 \geqslant 1$.*

**Corollary.** *Replacing $T_{4,i}$ and $T^1_{4,i}$ by $T^t_{4,i}$ and $T^{1t}_{4,i}$, respectively, in Theorem 18 (for $i = k, k_1, k_2, k_1 + k_2, k_1 + k_2 + 1$), the results hold.*

**Lemma 46.** $T_{4,k} \subseteq S_{4,k}$, $T^1_{4,k} \subseteq S^1_{4,k}$, $k \geqslant 1$.

**Corollary.** $T_{4,k} \subseteq S^t_{4,k}$, $T^{1t}_{4,k} \subseteq S^{1t}_{4,k}$, $k \geqslant 1$.

**Lemma 47.** *For any $p_1$, $p_2$, $k \geqslant 1$ and $p \geqslant 2$, there exists an automaton $M$ and $r \geqslant 2$ such that $|\mathcal{A}_j| = p_j$, $l_j = r_j = 1$, $j = 1, 2$, and $\exp_{p,k}$ is strongly computable in space $g^{(1)}_{r,k}$ by $M$.*

**Corollary.** *For any $k \geqslant 1$ and $p \geqslant 2$, there exists a unary automaton $M$ and $r \geqslant 2$ such that $l_j = r_j = 1$, $j = 1, 2$, and $\exp_{p,k}$ is strongly computable in space $g^{(1)}_{r,k}$ by $M$.*

**Lemma 48.** *If an $n$-ary function $f \in S_{4,k}$, $k \geqslant 1$, then for any $p_1, \ldots, p_{n+1} \geqslant 1$, there exists an automaton $M$ and $p \geqslant 2$ such that $|\mathcal{A}_j| = p_j$, $l_i = r_j = 1$, $j = 1, \ldots, n+1$, and $f$ is strongly computable in space $g^{(n)}_{p,k}$ by $M$.*

**Proof.** Similar to Lemma 18.

**Corollary.** *If an $n$-ary function $f \in S^1_{4,k}$, $k \geqslant 1$, then there exists a unary automaton $M$ and $p \geqslant 2$ such that $l_j = r_j = 1$, $j = 1, \ldots, n+1$, and $f$ is strongly computable in space $g^{(n)}_{p,k}$ by $M$.*

**Lemma 49.** (a) *If a n-ary function $f \in S_{4,k}$, $k \geqslant 1$, then there exists $p \geqslant 2$ such that*

$$f(x_1, \ldots, x_n) \leqslant \exp_{p,k+1}(\max(x_1, \ldots, x_n, 1))$$

*(when $f(x_1, \ldots, x_n)$ is defined).*

(b) *If an n-ary function $f \in S^1_{4,k}$, $k \geqslant 1$, then there exists $p \geqslant 2$ such that*

$$f(x_1, \ldots, x_n) \leqslant \exp_{p,k}(\max(x_1, \ldots, x_n, 1))$$

*(when $f(x_1, \ldots, x_n)$ is defined).*

**Proof.** Similar to Lemma 10.

**Theorem 19.** $S_{4,k} \subset S_{4,k+1}$, $S^1_{4,k} \subset S^1_{4,k+1}$, $k \geqslant 1$.

**Proof.** Similar to Theorem 14.

**Corollary.** $S^t_{4,k} \subset S^t_{4,k+1}$, $S^{1t}_{4,k} \subset S^{1t}_{4,k+1}$, $k \geqslant 1$.

**Theorem 20.** $S_4 = \bigcup_{k \geqslant 1} S_{4,k}$, $S^1_4 = \bigcup_{k \geqslant 1} S^1_{4,k}$

**Proof.** Similar to Theorem 15.

**Corollary** $S^t_4 = \bigcup_{k \geqslant 1} S^t_{4,k}$, $S^{1t}_4 = \bigcup_{k \geqslant 1} S^{1t}_{4,k}$.

Similar to Theorem 16, we have

**Theorem 21.** $S_3 \subset S_{4,1}$, $S^1_3 \subset S^1_{4,1}$.

**Corollary.** $S^t_3 \subset S^t_{4,1}$, $S^{1t}_3 \subset S^{1t}_{4,1}$.

**Theorem 22.** $S_{4,k} = S^1_{4,k+1}$, $k \geqslant 1$.

**Proof.** Similar to the proof of $S_0 = S^1_3$ in Theorem 11.

**Corollary 1.** $S^t_{4,k} = S^{1t}_{4,k+1}$, $k \geqslant 1$.

**Corollary 2.** $S_2 = S^1_{4,1}$.

**Corollary 3.** $S^t_2 = S^{1t}_{4,1}$.

**Theorem 23.** $T_{4,k} \subseteq S_{4,k} \subset T_{4,k+1}$, $T^1_{4,k} = S^1_{4,k}$, $k \geqslant 1$

**Proof.** Similar to Theorem 9.

**Corollary.** $T^t_{4,k} \subseteq S^t_{4,k} \subset T^t_{4,k+1}$, $T^{1t}_{4,k} = S^{1t}_{4,k}$, $k \geq 1$.

Above results can be summarized in following diagram:

$$
\begin{array}{ccccccccc}
T^1_{4,1} & & T^1_{4,2} & & T^1_{4,3} & & & & T^1_4 \\
\| & & \| & & \| & & & & \| \\
S_2 \subset T_{4,1} & \subseteq & S_{4,1} \subset T_{4,2} & \subseteq & S_{4,2} \subset T_{4,3} & \subseteq & \ldots & \subset & S_4 = T_4 \\
\| & & \| & & \| & & & & \| \\
S^1_{4,1} & & S^1_{4,2} & & S^1_{4,3} & & & & S^1_4
\end{array}
$$

For total functions, an analogous diagram can be made.

It immediately follows that the results of Theorem 18 and its Corollary still hold after replacing "$T$" by "$S$".

Similar to Lemma 39, we have

**Lemma 50.** *If* $g \in T_{4,k}$ *and* $k \geq 1$, *then* $\mathrm{TC}(g) \subseteq T_{4,k+1}$.

**Corollary 1.** *If* $g \in T^t_{4,k}$ *and* $k \geq 1$, *then* $\mathrm{TC}(g) \subseteq T^t_{4,k+1}$.

**Corollary 2.** *If* $g \in T^1_{4,k}$ *and* $k \geq 1$, *then* $\mathrm{TC}^1(g) \subseteq T^1_{4,k+1}$.

**Corollary 3.** *If* $g \in T^{1t}_{4,k}$ *and* $k \geq 1$, *then* $\mathrm{TC}^{1t}(g) \subseteq T^{1t}_{4,k}$.

Similar to Lemma 40, we have

**Lemma 51.** *The results of Lemma 50 and its corollaries still hold after replacing* "$T$" by "$S$".

**Theorem 24.**

(a)
$$\bigcup_{g \in T_{4,k}} \mathrm{TC}(g) \subseteq T_{4,k+1}, \qquad \bigcup_{g \in S_{4,k}} \mathrm{SC}(g) \subseteq S_{4,k+1},$$
$$\bigcup_{g \in T^t_{4,k}} \mathrm{TC}^t(g) \subseteq T^t_{4,k+1}, \qquad \bigcup_{g \in S^t_{4,k}} \mathrm{SC}^t(g) \subseteq S^t_{4,k+1}, \qquad k \geq 1.$$

(b)
$$\bigcup_{g \in T^1_{4,k}} \mathrm{TC}^1(g) = T^1_{4,k}, \qquad \bigcup_{g \in S^1_{4,k}} \mathrm{SC}^1(g) = S^1_{4,k},$$
$$\bigcup_{g \in T^{1t}_{4,k}} \mathrm{TC}^{1t}(g) = T^{1t}_{4,k}, \qquad \bigcup_{g \in S^{1t}_{4,k}} \mathrm{SC}^{1t}(g) = S^{1t}_{4,k}, \qquad k \geq 1.$$

**Proof.** Similar to Theorem 13, using Lemma 50 and its corollaries and Lemma 51.

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